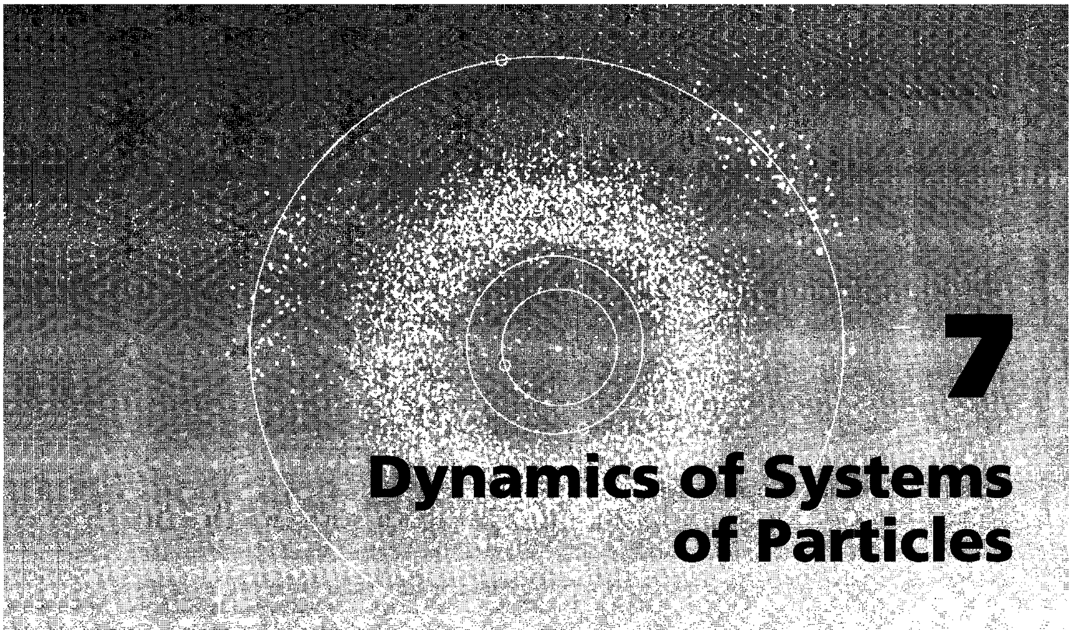

7 Dynamics of Systems of Particles

"Two equal bodies which are in direct impact with each other and have equal and opposite velocities before impact, rebound with velocities that are, apart from the sign, the same." "The sum of the products of the magnitudes of each hard body, multiplied by the square of the velocities, is always the same, before and after the collision."

— Christiaan Huygens, memoir, *De Motu Corporum ex mutuo impulsu Hypothesis*, composed in Paris, 5-Jan-1669, to Oldenburg, Secretary of the Royal Society

7.1 | Introduction: Center of Mass and Linear Momentum of a System

We now expand our study of mechanics of systems of many particles (two or more). These particles may or may not move independently of one another. Special systems, called *rigid bodies*, in which the relative positions of all the particles are fixed are taken up in the next two chapters. For the present, we develop some general theorems that apply to all systems. Then we apply them to some simple systems of free particles.

Our general system consists of n particles of masses $m_1, m_2, \ldots, m_n$ whose position vectors are, respectively, $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$. We define the *center of mass* of the system as the point whose position vector $\mathbf{r}_{cm}$ (Figure 7.1.1) is given by

$$\mathbf{r}_{cm} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_n\mathbf{r}_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_i m_i\mathbf{r}_i}{m} \tag{7.1.1}$$

where $m = \Sigma\, m_i$ is the total mass of the system. The definition in Equation 7.1.1 is equivalent to the three equations

$$x_{cm} = \frac{\sum_i m_i x_i}{m} \qquad y_{cm} = \frac{\sum_i m_i y_i}{m} \qquad z_{cm} = \frac{\sum_i m_i z_i}{m} \tag{7.1.2}$$

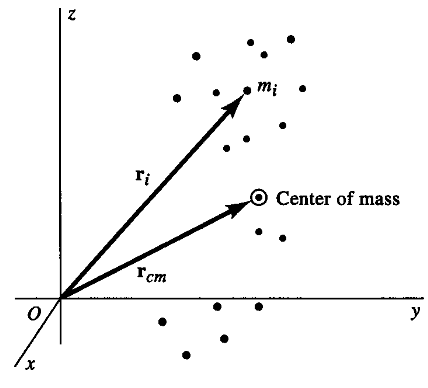

Figure 7.1.1 Center of mass of a system of particles.

We define the *linear momentum* $\mathbf{p}$ of the system as the vector sum of the linear momenta of the individual particles, namely,

$$\mathbf{p} = \sum_i \mathbf{p}_i = \sum_i m_i \mathbf{v}_i \tag{7.1.3}$$

On calculating $\dot{\mathbf{r}}_{cm} = \mathbf{v}_{cm}$ from Equation 7.1.1 and comparing with Equation 7.1.3, it follows that

$$\mathbf{p} = m\mathbf{v}_{cm} \tag{7.1.4}$$

that is, *the linear momentum of a system of particles is equal to the velocity of the center of mass multiplied by the total mass of the system.*

Suppose now that there are external forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_i, \ldots, \mathbf{F}_n$ acting on the respective particles. In addition, there may be internal forces of interaction between any two particles of the system. We denote these internal forces by $\mathbf{F}_{ij}$, meaning the force exerted on particle i by particle j, with the understanding that $\mathbf{F}_{ii} = 0$. The equation of motion of particle i is then

$$\mathbf{F}_i + \sum_{j=1}^{n} \mathbf{F}_{ij} = m_i \ddot{\mathbf{r}}_i = \dot{\mathbf{p}}_i \tag{7.1.5}$$

where $\mathbf{F}_i$ means the total external force acting on particle i. The second term in Equation 7.1.5 represents the vector sum of all the internal forces exerted on particle i by all other particles of the system. Adding Equation 7.1.5 for the n particles, we have

$$\sum_{i=1}^{n} \mathbf{F}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{F}_{ij} = \sum_{i=1}^{n} \dot{\mathbf{p}}_i \tag{7.1.6}$$

In the double summation in Equation 7.1.6, for every force $\mathbf{F}_{ij}$ there is also a force $\mathbf{F}_{ji}$, and these two forces are equal and opposite

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji} \tag{7.1.7}$$

from the law of action and reaction, Newton's third law. Consequently, the internal forces cancel in pairs, and the double sum vanishes. We can, therefore, write Equation 7.1.7 in the following way:

$$\sum_i \mathbf{F}_i = \dot{\mathbf{p}} = m\mathbf{a}_{cm} \tag{7.1.8}$$

In words: *The acceleration of the center of mass of a system of particles is the same as that of a single particle having a mass equal to the total mass of the system and acted on by the sum of the external forces.*

Consider, for example, a swarm of particles moving in a *uniform* gravitational field. Then, because $\mathbf{F}_i = m_i\mathbf{g}$ for each particle,

$$\sum_i \mathbf{F}_i = \sum m_i\mathbf{g} = m\mathbf{g} \tag{7.1.9}$$

The last step follows from the fact that $\mathbf{g}$ is constant. Hence,

$$\mathbf{a}_{cm} = \mathbf{g} \tag{7.1.10}$$

This is the same as the equation for a single particle or projectile. Thus, the center of mass of the shrapnel from an artillery shell that has burst in midair follows the same parabolic path that the shell would have taken had it not burst (until any of the pieces strikes something).

In the special case in which no external forces are acting on a system (or if $\Sigma\,\mathbf{F}_i = 0$), then $\mathbf{a}_{cm} = 0$ and $\mathbf{v}_{cm} =$ constant; thus, the linear momentum of the system remains constant:

$$\sum_i \mathbf{p}_i = \mathbf{p} = m\mathbf{v}_{cm} = \text{constant} \tag{7.1.11}$$

This is the *principle of conservation of linear momentum.* In Newtonian mechanics the constancy of the linear momentum of an isolated system is directly related to, and is in fact a consequence of, the third law. But even in those cases in which the forces between particles do not directly obey the law of action and reaction, such as the magnetic forces between moving charges, the principle of conservation of linear momentum still holds when due account is taken of the total linear momentum of the particles and the electromagnetic field.[1]

EXAMPLE 7.1.1

At some point in its trajectory a ballistic missile of mass m breaks into three fragments of mass $m/3$ each. One of the fragments continues on with an initial velocity of one-half the velocity $\mathbf{v}_0$ of the missile just before breakup. The other two pieces go off at right angles to each other with equal speeds. Find the initial speeds of the latter two fragments in terms of v_0.

Solution:

At the point of breakup, conservation of linear momentum is expressed as

$$m\mathbf{v}_{cm} = m\mathbf{v}_0 = \frac{m}{3}\mathbf{v}_1 + \frac{m}{3}\mathbf{v}_2 + \frac{m}{3}\mathbf{v}_3$$

[1] See, for example, P. M. Fishbane, S. Gasiorowicz, S. T. Thornton, *Physics for Scientists and Engineers.* Prentice-Hall, Englewood Cliffs, NJ, 1993.

The given conditions are: $\mathbf{v}_1 = \mathbf{v}_0/2$, $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$, and $v_2 = v_3$. From the first we get, on cancellation of the m's, $3\mathbf{v}_0 = (\mathbf{v}_0/2) + \mathbf{v}_2 + \mathbf{v}_3$, or

$$\tfrac{5}{2}\mathbf{v}_0 = \mathbf{v}_2 + \mathbf{v}_3$$

Taking the dot product of each side with itself, we have

$$\tfrac{25}{4}v_0^2 = (\mathbf{v}_2 + \mathbf{v}_3) \cdot (\mathbf{v}_2 + \mathbf{v}_3) = v_2^2 + 2\mathbf{v}_2 \cdot \mathbf{v}_3 + v_3^2 = 2v_2^2$$

Therefore,

$$v_2 = v_3 = \frac{5}{2\sqrt{2}}v_0 = 1.77v_0$$

7.2 | Angular Momentum and Kinetic Energy of a System

We previously stated that the angular momentum of a single particle is defined as the cross product $\mathbf{r} \times m\mathbf{v}$. The angular momentum $\mathbf{L}$ of a system of particles is defined accordingly, as the vector sum of the individual angular momenta, namely,

$$\mathbf{L} = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i\mathbf{v}_i) \tag{7.2.1}$$

Let us calculate the time derivative of the angular momentum. Using the rule for differentiating the cross product, we find

$$\frac{d\mathbf{L}}{dt} = \sum_{i=1}^{n} (\mathbf{v}_i \times m_i\mathbf{v}_i) + \sum_{i=1}^{n} (\mathbf{r}_i \times m_i\mathbf{a}_i) \tag{7.2.2}$$

Now the first term on the right vanishes, because, $\mathbf{v}_i \times \mathbf{v}_i = 0$ and, because $m_i\mathbf{a}_i$ is equal to the total force acting on particle i, we can write

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \sum_{i=1}^{n}\left[\mathbf{r}_i \times \left(\mathbf{F}_i + \sum_{j=1}^{n}\mathbf{F}_{ij}\right)\right] \\ &= \sum_{i=1}^{n}\mathbf{r}_i \times \mathbf{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{r}_i \times \mathbf{F}_{ij}\end{aligned} \tag{7.2.3}$$

where, as in Section 7.1, $\mathbf{F}_i$ denotes the total external force on particle i, and $\mathbf{F}_{ij}$ denotes the (internal) force exerted on particle i by any other particle j. Now the double summation on the right consists of pairs of terms of the form

$$(\mathbf{r}_i \times \mathbf{F}_{ij}) + (\mathbf{r}_j \times \mathbf{F}_{ji}) \tag{7.2.4}$$

Denoting the vector displacement of particle j relative to particle i by $\mathbf{r}_{ij}$, we see from the triangle shown in Figure 7.2.1 that

$$\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i \tag{7.2.5}$$

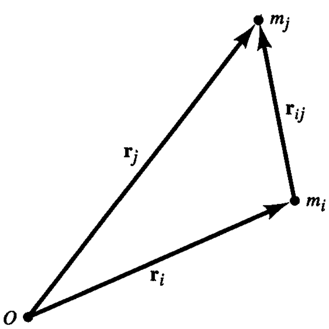

Figure 7.2.1 Definition of the vector $\mathbf{r}_{ij}$.

Therefore, because $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$, expression 7.2.4 reduces to

$$-\mathbf{r}_{ij} \times \mathbf{F}_{ij} \tag{7.2.6}$$

which clearly vanishes if the internal forces are central, that is, if they act along the lines connecting pairs of particles. Hence, the double sum in Equation 7.2.3 vanishes. Now the cross product $\mathbf{r}_i \times \mathbf{F}_i$ is the moment of the external force $\mathbf{F}_i$. The sum $\Sigma\, \mathbf{r}_i \times \mathbf{F}_i$ is, therefore, the total moment of all the external forces acting on the system. If we denote the total external torque, or moment of force, by $\mathbf{N}$, Equation 7.2.3 takes the form

$$\frac{d\mathbf{L}}{dt} = \mathbf{N} \tag{7.2.7}$$

That is, *the time rate of change of the angular momentum of a system is equal to the total moment of all the external forces acting on the system.*

If a system is isolated, then $\mathbf{N} = 0$, and the angular momentum remains constant in both magnitude and direction:

$$\mathbf{L} = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i = \text{constant vector} \tag{7.2.8}$$

This is a statement of the *principle of conservation of angular momentum.* It is a generalization for a single particle in a central field. Like the constancy of linear momentum discussed in the preceding section, the angular momentum of an isolated system is also constant in the case of a system of moving charges when the angular momentum of the electromagnetic field is considered.[2]

It is sometimes convenient to express the angular momentum in terms of the motion of the center of mass. As shown in Figure 7.2.2, we can express each position vector $\mathbf{r}_i$ in the form

$$\mathbf{r}_i = \mathbf{r}_{cm} + \bar{\mathbf{r}}_i \tag{7.2.9}$$

where $\bar{\mathbf{r}}_i$ is the position of particle i relative to the center of mass. Taking the derivative with respect to t, we have

$$\mathbf{v}_i = \mathbf{v}_{cm} + \bar{\mathbf{v}}_i \tag{7.2.10}$$

[2] See footnote 1.

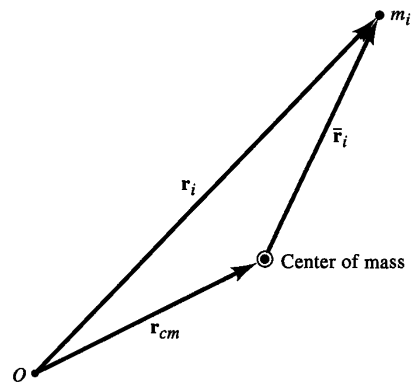

Figure 7.2.2 Definition of the vector $\bar{\mathbf{r}}_i$.

Here $\mathbf{v}_{cm}$ is the velocity of the center of mass and $\bar{\mathbf{v}}_i$ is the velocity of particle i relative to the center of mass. The expression for $\mathbf{L}$ can, therefore, be written

$$\begin{aligned}
\mathbf{L} &= \sum_i (\mathbf{r}_{cm} + \bar{\mathbf{r}}_i) \times m_i(\mathbf{v}_{cm} + \bar{\mathbf{v}}_i) \\
&= \sum_i (\mathbf{r}_{cm} \times m_i\mathbf{v}_{cm}) + \sum_i (\mathbf{r}_{cm} \times m_i\bar{\mathbf{v}}_i) \\
&\quad + \sum_i (\bar{\mathbf{r}}_i \times m_i\mathbf{v}_{cm}) + \sum_i (\bar{\mathbf{r}}_i \times m_i\bar{\mathbf{v}}_i) \\
&= \mathbf{r}_{cm} \times \left(\sum_i m_i\right)\mathbf{v}_{cm} + \mathbf{r}_{cm} \times \sum_i m_i\bar{\mathbf{v}}_i \\
&\quad + \left(\sum_i m_i\bar{\mathbf{r}}_i\right) \times \mathbf{v}_{cm} + \sum_i (\bar{\mathbf{r}}_i \times m_i\bar{\mathbf{v}}_i)
\end{aligned} \tag{7.2.11}$$

Now, from Equation 7.2.9, we have

$$\sum_i m_i\bar{\mathbf{r}}_i = \sum_i m_i(\mathbf{r}_i - \mathbf{r}_{cm}) = \sum_i m_i\mathbf{r}_i - m\mathbf{r}_{cm} = 0 \tag{7.2.12}$$

Similarly, we obtain

$$\sum_i m_i\bar{\mathbf{v}}_i = \sum_i m_i\mathbf{v}_i - m\mathbf{v}_{cm} = 0 \tag{7.2.13}$$

by differentiation with respect to t. (These two equations merely state that the position and velocity of the center of mass, relative to the center of mass, are both zero.) Consequently, the second and third summations in the expansion of $\mathbf{L}$ vanish, and we can write

$$\mathbf{L} = \mathbf{r}_{cm} \times m\mathbf{v}_{cm} + \sum_i \bar{\mathbf{r}}_i \times m_i\bar{\mathbf{v}}_i \tag{7.2.14}$$

expressing the angular momentum of a system in terms of an "orbital" part (motion of the center of mass) and a "spin" part (motion about the center of mass).

EXAMPLE 7.2.1

A long, thin rod of length l and mass m hangs from a pivot point about which it is free to swing in a vertical plane like a simple pendulum. Calculate the total angular momentum of the rod as a function of its instantaneous angular velocity ω. Show that the theorem represented by Equation 7.2.14 is true by comparing the angular momentum obtained using that theorem to that obtained by direct calculation.

Solution:

The rod is shown in Figure 7.2.3a. First we calculate the angular momentum $\mathbf{L}_{cm}$ of the center of mass of the rod about the pivot point. Because the velocity $\mathbf{v}_{cm}$ of the center of mass is always perpendicular to the radius vector $\mathbf{r}$ denoting its location relative to the pivot point, the sine of the angle between those two vectors is unity. Thus, the magnitude of $\mathbf{L}_{cm}$ is given by

$$L_{cm} = \frac{l}{2}p_{cm} = m\frac{l}{2}v_{cm} = m\frac{l}{2}\left(\frac{l}{2}\omega\right) = \tfrac{1}{4}ml^2\omega$$

Figure 7.2.3b depicts the motion of the rod as seen from the perspective of its center of mass. The angular momentum dL_{rel} of two small mass elements, each of size dm symmetrically disposed about the center of mass of the rod, is given by

$$dL_{rel} = 2rdp = 2rv\,dm = 2r(r\omega)\lambda\,dr$$

where λ is the mass per unit length of the rod. The total relative angular momentum is obtained by integrating this expression from $r = 0$ to $r = l/2$.

$$L_{rel} = 2\lambda\omega\int_0^{l/2} r^2 dr = \tfrac{1}{12}(\lambda l)l^2\omega = \left(\tfrac{1}{12}ml^2\right)\omega$$

We can see in the preceding equation that the angular momentum of the rod about its center of mass is directly proportional to the angular velocity ω of the rod. The constant of proportionality $ml^2/12$ is called the *moment of inertia* I_{cm} of the rod about its center of mass. Moment of inertia plays a role in rotational motion similar to that of inertial mass in translational motion as we shall see in the next chapter.

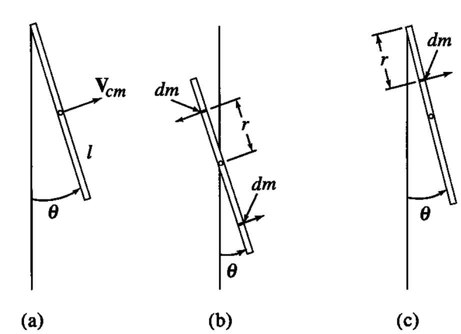

Figure 7.2.3 Rod of mass m and length l free to swing in a vertical plane about a fixed pivot.

Finally, the total angular momentum of the rod is

$$L_{tot} = L_{cm} + L_{rel} = \tfrac{1}{3}ml^2\omega$$

Again, the total angular momentum of the rod is directly proportional to the angular velocity of the rod. Here, though, the constant of proportionality is the moment of inertia of the rod about the pivot point at the end of the rod. This moment of inertia is larger than that about the center of mass. The reason is that more of the mass of the rod is distributed farther away from its end than from its center, thus, making it more difficult to rotate a rod about an end.

The total angular momentum can also be obtained by integrating down the rod, starting from the pivot point, to obtain the contribution from each mass element dm, as shown in Figure 7.2.3c

$$dL_{tot} = r\,dp = r(v\,dm) = r(r\omega)\lambda\,dr$$

$$L_{tot} = \lambda\omega\int_0^l r^2 dr = \tfrac{1}{3}ml^2\omega$$

And, indeed, the two methods yield the same result.

Kinetic Energy of a System

The total kinetic energy T of a system of particles is given by the sum of the individual energies, namely,

$$T = \sum_i \tfrac{1}{2}m_i v_i^2 = \sum_i \tfrac{1}{2}m_i(\mathbf{v}_i \cdot \mathbf{v}_i) \tag{7.2.15}$$

As before, we can express the velocities relative to the mass center giving

$$\begin{aligned} T &= \sum_i \tfrac{1}{2}m_i(\mathbf{v}_{cm} + \bar{\mathbf{v}}_i)\cdot(\mathbf{v}_{cm} + \bar{\mathbf{v}}_i) \\ &= \sum_i \tfrac{1}{2}m_i v_{cm}^2 + \sum_i m_i(\mathbf{v}_{cm} \cdot \bar{\mathbf{v}}_i) + \sum_i \tfrac{1}{2}m_i \bar{v}_i^2 \\ &= \tfrac{1}{2}v_{cm}^2 \sum_i m_i + \mathbf{v}_{cm} \cdot \sum_i m_i \bar{\mathbf{v}}_i + \sum_i \tfrac{1}{2}m_i \bar{v}_i^2 \end{aligned} \tag{7.2.16}$$

Because the second summation $\sum_i m_i \bar{\mathbf{v}}_i$ vanishes, we can express the kinetic energy as follows:

$$T = \tfrac{1}{2}mv_{cm}^2 + \sum_i \tfrac{1}{2}m_i \bar{v}_i^2 \tag{7.2.17}$$

The first term is the kinetic energy of translation of the whole system, and the second is the kinetic energy of motion relative to the mass center.

The separation of angular momentum and kinetic energy into a center-of-mass part and a relative-to-center-of-mass part finds important applications in atomic and molecular physics and in astrophysics. We find the preceding two theorems useful in the study of rigid bodies in the following chapters.

EXAMPLE 7.2.2

Calculate the total kinetic energy of the rod of Example 7.2.1. Use the theorem represented by Equation 7.2.17. As in Example 7.2.1, show that the total energy obtained for the rod according to this theorem is equivalent to that obtained by direct calculation.

Solution:

The translational kinetic energy of the center of mass of the rod is

$$T_{cm} = \tfrac{1}{2} m\mathbf{v}_{cm} \cdot \mathbf{v}_{cm} = \tfrac{1}{2} m\left(\frac{l}{2}\omega\right)^2 = \tfrac{1}{8} ml^2\omega^2$$

The kinetic energy of two equal mass elements dm symetrically disposed about the center of mass is

$$dT_{rel} = \tfrac{1}{2}(2dm)\mathbf{v} \cdot \mathbf{v} = \lambda\, dr(r\omega)^2 = \lambda\omega^2 r^2 dr$$

where λ, again, is the mass per unit length of the rod. The total energy relative to the center of mass can be obtained by integrating the preceding expression from $r = 0$ to $r = l/2$.

$$T_{rel} = \lambda\omega^2 \int_0^{l/2} r^2 dr = \tfrac{1}{24}\lambda\omega^2 l^3 = \tfrac{1}{2}\left(\tfrac{1}{12} ml^2\right)\omega^2 = \tfrac{1}{2} I_{cm}\omega^2$$

(**Note:** As in Example 7.2.1, the moment of inertia term I_{cm} appears as the constant of proportionality to ω^2 in the previous expression for the rotational kinetic energy of the rod about its center of mass. Again, the moment of inertia term that occurs in the expression for rotational kinetic energy can be seen to be completely analogous to the inertial mass term in an expression for the translational kinetic energy of a particle.)

The total kinetic energy of the rod is then

$$T = T_{cm} + T_{rel} = \tfrac{1}{8} ml^2\omega^2 + \tfrac{1}{24} ml^2\omega^2 = \tfrac{1}{2}\left(\tfrac{1}{3} ml^2\right)\omega^2 = \tfrac{1}{2} I\omega^2$$

where we have expressed the final result in terms of the total moment of inertia of the rod about its endpoint, exactly as in Example 7.2.1.

We leave it as an exercise for the reader to calculate the kinetic energy directly and show that it is equal to the value obtained previously. The calculation proceeds in a fashion completely analogous to that in Example 7.2.1.

7.3 Motion of Two Interacting Bodies: The Reduced Mass

Let us consider the motion of a system consisting of two bodies, treated here as particles, that interact with each other by a central force. We assume the system is isolated, and, hence, the center of mass moves with constant velocity. For simplicity, we take the center of mass as the origin. We have then

$$m_1\bar{\mathbf{r}}_1 + m_2\bar{\mathbf{r}}_2 = 0 \tag{7.3.1}$$

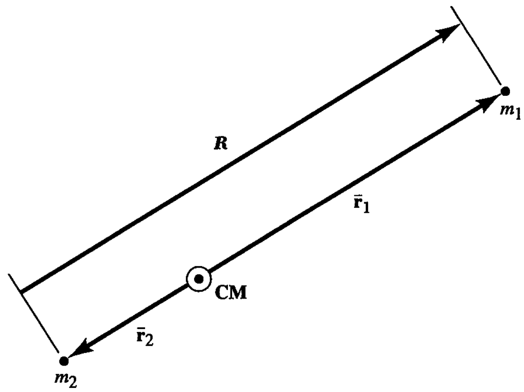

Figure 7.3.1 The relative position vector $\mathbf{R}$ for the two-body problem.

where, as shown in Figure 7.3.1, the vectors $\bar{\mathbf{r}}_1$ and $\bar{\mathbf{r}}_2$ represent the positions of the particles m_1 and m_2, respectively, relative to the center of mass. Now, if $\mathbf{R}$ is the position vector of particle 1 relative to particle 2, then

$$\mathbf{R} = \bar{\mathbf{r}}_1 - \bar{\mathbf{r}}_2 = \bar{\mathbf{r}}_1\left(1 + \frac{m_1}{m_2}\right) \tag{7.3.2}$$

The last step follows from Equation 7.3.1.

The differential equation of motion of particle 1 relative to the center of mass is

$$m_1 \frac{d^2\bar{\mathbf{r}}_1}{dt^2} = \mathbf{F}_1 = f(R)\frac{\mathbf{R}}{R} \tag{7.3.3}$$

in which $|f(R)|$ is the magnitude of the mutual force between the two particles. By using Equation 7.3.2, we can write

$$\mu \frac{d^2\mathbf{R}}{dt^2} = f(R)\frac{\mathbf{R}}{R} \tag{7.3.4}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{7.3.5}$$

The quantity μ is called the *reduced mass.* The new equation of motion (Equation 7.3.4) gives the motion of particle 1 relative to particle 2, and an exactly similar equation gives the motion of particle 2 relative to particle 1. This equation is precisely the same as the ordinary equation of motion of a single particle of mass μ moving in a central field of force given by $f(R)$. Thus, the fact that both particles are moving relative to the center of mass is automatically accounted for by replacing m_1 by the reduced mass μ. If the bodies are of equal mass m, then $\mu = m/2$. On the other hand, if m_2 is very much greater than m_1, so that m_1/m_2 is very small, then μ is nearly equal to m_1.

For two bodies attracting each other by gravitation

$$f(R) = -\frac{Gm_1m_2}{R^2} \tag{7.3.6}$$

In this case the equation of motion is

$$\mu\ddot{\mathbf{R}} = -\frac{Gm_1m_2}{R^2}\mathbf{e}_R \tag{7.3.7}$$

or, equivalently,

$$m_1\ddot{\mathbf{R}} = -\frac{G(m_1+m_2)m_1}{R^2}\mathbf{e}_R \tag{7.3.8}$$

where $\mathbf{e}_R = \mathbf{R}/R$ is a unit vector in the direction of $\mathbf{R}$.

In Section 6.6 we derived an equation giving the periodic time of orbital motion of a planet of mass m moving in the Sun's gravitational field, namely, $\tau = 2\pi\,(GM_\odot)^{-1/2}a^{3/2}$, where $M_\odot$ is the Sun's mass and a is the semimajor axis of the elliptical orbit of the planet about the Sun. In that derivation we assumed that the Sun was stationary, with the origin of our coordinate system at the center of the Sun. To account for the Sun's motion about the common center of mass, the correct equation is Equation 7.3.8 in which $m = m_1$ and $M_\odot = m_2$. The constant k, which was taken to be $GM_\odot m$ in the earlier treatment, should be replaced by $G(M_\odot + m)m$ so that the correct equation for the period is

$$\tau = 2\pi\,[G(M_\odot + m)]^{-1/2}a^{3/2} \tag{7.3.9a}$$

or, for any two-body system held together by gravity, the orbital period is

$$\tau = 2\pi\,[G(m_1+m_2)]^{-1/2}a^{3/2} \tag{7.3.9b}$$

If m_1 and m_2 are expressed in units of the Sun's mass and a is in astronomical units (the mean distance from Earth to the Sun), then the orbital period in years is given by

$$\tau = (m_1+m_2)^{-1/2}a^{3/2} \tag{7.3.9c}$$

For most planets in our solar system, the added mass term in the preceding expression for the period makes very little difference—Earth's mass is only 1/330,000 the Sun's mass. The most massive planet, Jupiter, has a mass of about 1/1000 the mass of the Sun, so the effect of the reduced-mass formula is to change the earlier calculation in the ratio $(1.001)^{-1/2} = 0.9995$ for the period of Jupiter's revolution about the Sun.

Binary Stars: White Dwarfs and Black Holes

About half of all the stars in the galaxy in the vicinity of the Sun are binary, or double; that is, they occur in pairs held together by their mutual gravitational attraction, with each member of the pair revolving about their common center of mass. From the preceding analysis we can infer that either member of a binary system revolves about the other in an elliptical orbit for which the orbiting period is given by Equations 7.3.9b and c, where a is the semimajor axis of the ellipse and m_1 and m_2 are the masses of the two stars. Values of a for known binary systems range from the very least (*contact binaries* in which the stars touch each other) to values so large that the period is measured in millions of years. A typical example is the brightest star in the night sky, Sirius, which consists of a very luminous star with a mass of 2.1 $M_\odot$ and a very small dim star, called a *white dwarf*, which can only be seen in large telescopes. The mass of this small companion is 1.05 $M_\odot$, but its size is roughly that of a large planet, so its density is extremely large (30,000 times the density of water). The value of a for the Sirius system is approximately 20 AU (about the distance from the Sun to the planet Uranus), and the period, as calculated from Equation 7.3.9c, should be about

$$\tau = (2.1 + 1.05)^{-1/2}(20)^{3/2} \text{ years} = 50 \text{ years}$$

which is what it is observed to be.

A binary system that is believed to harbor a black hole as one of its components is the x-ray source known as Cygnus X-1.[3] The visible component is the normal star HDE 226868. Spectroscopic observation of the optical light from this star indicates that the period and semimajor axis of the orbit are 5.6 days and about 30×10^6 km, respectively. The optically invisible companion is the source of an x-ray flux that exhibits fluctuations that vary as rapidly as a millisecond, indicating that it can be no larger than 300 km across. These observations, as well as a number of others, indicate that the mass of HDE 226868 is at least as large as 20 $M_\odot$, while that of its companion is probably as large as 16 $M_\odot$ but surely exceeds 7 $M_\odot$. It is difficult to conclude that this compact, massive object could be anything other than a *black hole*. Black holes are objects that contain so much mass within a given radius[4] that nothing, not even light, can escape their gravitational field. If black holes are located in binary systems, however, mass can "leak over" from the large companion star and form an accretion disk about the black hole. As the matter in this disk orbits the black hole, it can lose energy by frictional heating and crash down into it, ultimately heating to temperatures well in excess of tens of millions of degrees. X-rays are emitted by this hot matter before it falls completely into the hole (Figure 7.3.2). Black holes are predicted mathematically by the general theory of relativity, and unequivocal proof of their existence would constitute a milestone in astrophysics.

[3] A. P. Cowley, *Ann. Rev. Astron. Astrophys.* 30, 287 (1992).

[4] According to the theory of general relatively, a nonrotating, spherically symmetric body of mass m becomes a Schwarzschild black hole if it is compressed to a radius r_s, Schwarzschild radius, where

$$r_s = \frac{2Gm}{c^2}$$

in which c is the speed of light. The Earth would become a black hole if compressed to the size of a small marble; the Sun would become one if compressed to a radius of about 3 km, much smaller than the white dwarf companion of Sirius.

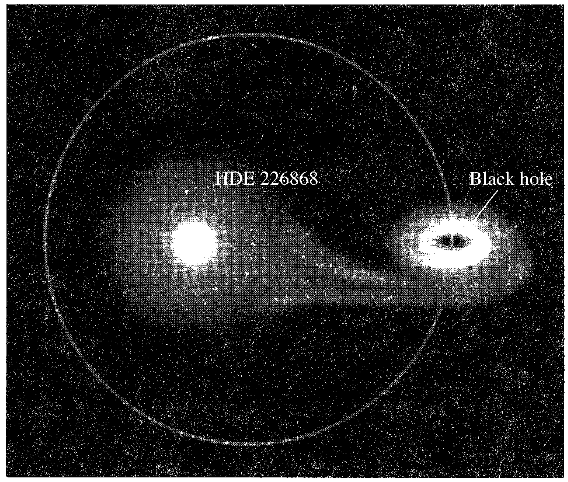

Figure 7.3.2 The Cygnus X-1 System.

EXAMPLE 7.3.1

A certain binary star system is observed to be both eclipsing and spectroscopic. This means that the system is seen from Earth with its orbital plane edge-on and that the orbital velocities v_1 and v_2 of the two stars that constitute the system can be determined from Doppler shift measurements of observed spectral lines. You don't need to understand the details of this last statement. The important point is that we know the orbital velocities. They are, in appropriate units, $v_1 = 1.257$ AU/year and $v_2 = 5.027$ AU/year. The period of revolution of each star about its center of mass is $\tau = 5$ years. (That can be ascertained from the observed frequency of eclipses.) Calculate the mass (in solar mass units $M_\odot$) of each star. Assume circular orbits.

Solution:

The radius of the orbit of each star about their common center of mass can be calculated from its velocity and period

$$r_1 = \frac{1}{2\pi} v_1 \tau = 1 \text{ AU} \qquad r_2 = \frac{1}{2\pi} v_2 \tau = 4 \text{ AU}$$

Thus, the semimajor axis a of the orbit is

$$a = r_1 + r_2 = 5 \text{ AU}$$

The sum of the masses can be obtained from Equation 7.3.9c

$$m_1 + m_2 = \frac{a^3}{\tau^2} = 5\, M_\odot$$

The ratio of the two masses can be determined by differentiating Equation 7.3.1

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = 0 \qquad \frac{m_2}{m_1} = \left|\frac{\mathbf{v}_1}{\mathbf{v}_2}\right| = \frac{1}{4}$$

Combining these last two expressions yields the values for each mass, $m_1 = 4\ M_\odot$ and $m_2 = 1\ M_\odot$.

*7.4 The Restricted Three-Body Problem[5]

In Chapter 6, we considered the motion of a single particle subject to a central force. The motion of a planet in the gravitational field of the Sun is well described by such a theory because the mass of the Sun is so large compared with that of a planet that its own motion can be ignored. In the previous section, we relaxed this condition and found that we could still apply the techniques of Newtonian analysis to this more general case and find an analytic solution for their motion. If we add just one more, third body, however, the problem becomes completely intractable. The general three-body problem, namely the calculation of the motion of three bodies of different masses, initial positions, and velocities, subject to the combined gravitational field of the others, confounded some of the greatest minds in the post-Newtonian era. It is not possible to solve this problem analytically because of insurmountable mathematical difficulties. Indeed, the problem is described by a system of nine second-order differential equations: three bodies moving in three dimensions. Even after a mathematical reduction accomplished by a judicious choice of coordinate system and by invoking laws of conservation to find invariants of the motion, the problem continues to defy assault by modern analytic techniques.

Fortunately, it is possible to solve a simplified case of the general problem that nonetheless describes a wide variety of phenomena. This special case is called the *restricted three-body problem.* The simplifications involved are both physical and mathematical: We assume that two of the bodies (called the *primaries*[6]) are much more massive than the third body (called the *tertiary*) and that they move in a plane—in circular orbits about their center of mass. The tertiary has a negligible mass compared with either of the primaries, moves in their orbital plane, and exerts no gravitational influence on either of them.

No physical system meets these requirements exactly. The tertiary always perturbs the orbits of the primaries. Perfectly circular orbits never occur, although most of the orbits of bodies in the solar system come very close—with the exception of comets. The orbit of the tertiary is almost never coplanar with those of the primaries, although deviations from coplanarity are often quite small. Gravitational systems with a dominant central mass exhibit a remarkable propensity for coplanarity. Again, disregarding the comets, the remaining members of the solar system exhibit a high degree of coplanarity, as do the individual systems of the large Jovian planets and their assemblage of moons.

[5]Our analysis of the restricted three-body problem is based on P. Hellings, *Astrophysics with a PC, An Introduction to Computational Astrophysics*, Willman-Bell, Inc., Richmond, VA (1994). Also, for an even more in-depth analysis see V. Szebehely, *Theory of Orbits*, Academic Press, New York (1967).

[6]Usually, the most massive of the pair is called the primary and the least massive is called the secondary. Here, we lump them together as the two primaries because their motion is only incidental to our main interest—the motion of the third body.

The restricted three-body problem serves as an excellent model for calculating the orbital motion of a small tertiary in the gravitational field of the other two. It is fairly easy to see two possible solutions depicting two extreme situations. One occurs when the tertiary more or less orbits the center of mass of the other two at such a remote distance that the two primaries appear to blur together as a single gravitational source. A second occurs when the tertiary is bound so closely to one of the primaries that it orbits it in Keplerian fashion, seemingly oblivious to the presence of the second primary. Both of these possibilities are realized in nature. In this section, however, we attempt to find a third, not so obvious, "stationary" solution; that is, one in which the tertiary is "held fixed" by the other two and partakes of their overall rotational motion. In other words, it remains more or less at rest relative to the two primaries; the orientation in space of the entire system rotates with a constant angular speed, but its relative configuration remains fixed in time. The great 18th-century mathematician Joseph-Louis Lagrange (1736–1813) solved this problem and showed that such orbits are possible.

Equations of Motion for the Restricted Three-Body Problem

The restricted problem is a two-dimensional one: All orbits lie within a single, fixed plane in space. The orbit of each of the two primaries is a circle with common angular velocity ω about their center of mass. We assume that the center of mass of the two primaries remains fixed in space and that the rotational sense of their orbital motion viewed from above is counterclockwise as shown in Figure 7.4.1

We designate M_1 the mass of the most massive primary, M_2 the mass of the least massive one, and m the small mass of the tertiary whose orbit we wish to calculate. We choose a coordinate system x'-y' that rotates with the two primaries and whose origin is their center of mass. We let the $+x$-axis lie along the direction toward the most massive primary M_1. The radii of the circular orbits of M_1 and M_2 are designated a and b, respectively. These distances remain fixed along the x'-axis in the rotating coordinate system.

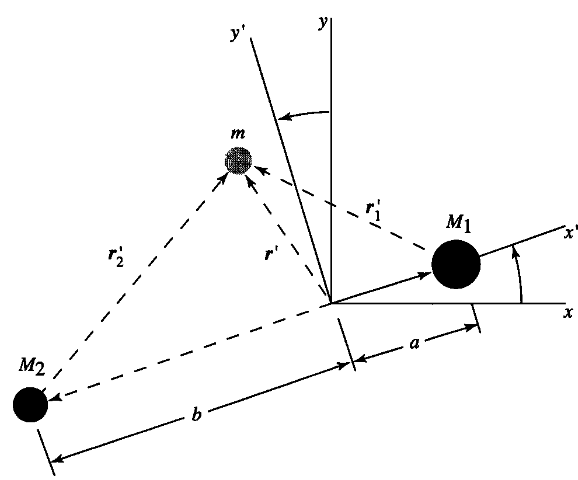

Figure 7.4.1 Coordinate system for restricted three-body problem.

Letting the coordinates of the tertiary be (x', y'), the distance between it and each of the two primaries is

$$r_1' = \sqrt{(x'-a)^2 + y'^2} \tag{7.4.1a}$$

$$r_2' = \sqrt{(x'+b)^2 + y'^2} \tag{7.4.1b}$$

The net gravitational force exerted on m (see Equation 6.1.1) is thus

$$\mathbf{F} = -m\frac{GM_1}{r_1'^2}\left(\frac{\mathbf{r}_1'}{r_1'}\right) - m\frac{GM_2}{r_2'^2}\left(\frac{\mathbf{r}_2'}{r_2'}\right) \tag{7.4.2}$$

where $\mathbf{r}_1'$ and $\mathbf{r}_2'$ are the vector positions of m with respect of M_1 and M_2. This force is the only real one that acts on m, but because we have effectively nullified the motion of the two primaries by choosing to calculate the motion in a frame of reference that rotates with them, we must include the effect of the noninertial forces that are introduced as a result of this choice.

The general equation of motion for a particle in a rotating frame of reference was given by Equation 5.3.2. Because the origin of the rotating coordinate system remains fixed in space, $\mathbf{A}_0 = 0$, and because the rate of rotation is a constant, $\dot{\boldsymbol{\omega}} = 0$ and Equation 5.3.2 takes the form

$$\mathbf{F}' = m\mathbf{a}' = \mathbf{F} - 2m\boldsymbol{\omega} \times \mathbf{v}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \tag{7.4.3}$$

Because m is common to all terms in Equation 7.4.3, we can rewrite it in terms of accelerations as

$$\mathbf{a}' = \frac{\mathbf{F}}{m} - 2\boldsymbol{\omega} \times \mathbf{v}' - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \tag{7.4.4}$$

We are now in a position to calculate the later two noninertial accelerations in Equation 7.4.4—the Coriolis and centrifugal accelerations

$$2\boldsymbol{\omega} \times \mathbf{v}' = 2\omega\mathbf{k}' \times (\mathbf{i}'\dot{x}' + \mathbf{j}'\dot{y}') = -\mathbf{i}'2\omega\dot{y}' + \mathbf{j}'2\omega\dot{x}' \tag{7.4.5}$$

and

$$\begin{aligned}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') &= \omega\mathbf{k}' \times [\omega\mathbf{k}' \times (\mathbf{i}'x' + \mathbf{j}'y')] \\ &= -\mathbf{i}'\omega^2 x' - \mathbf{j}'\omega^2 y'\end{aligned} \tag{7.4.6}$$

We now insert Equations 7.4.1a and b, 7.4.2, 7.4.5, and 7.4.6 into 7.4.4 to obtain the equations of motion of mass m in the x' and y' coordinates

$$\begin{aligned}\ddot{x}' = &-GM_1\frac{x'-a}{[(x'-a)^2 + y'^2]^{3/2}} - GM_2\frac{x'+b}{[(x'+b)^2 + y'^2]^{3/2}} \\ &+ \omega^2 x' + 2\omega\dot{y}'\end{aligned} \tag{7.4.7a}$$

$$\begin{aligned}\ddot{y}' = &-GM_1\frac{y'}{[(x'-a)^2 + y'^2]^{3/2}} - GM_2\frac{y'}{[(x'+b)^2 + y'^2]^{3/2}} \\ &+ \omega^2 y' - 2\omega\dot{x}'\end{aligned} \tag{7.4.7b}$$

The Effective Potential: The Five Lagrangian Points

Before solving Equations 7.4.7a and b, we would like to speculate about the possible solutions that we might obtain. Toward this end, we note that the first three terms in each of those equations can be expressed as the gradient of an effective potential function, $V(r')$ in polar coordinates

$$V(r') = -\frac{GM_1}{|\mathbf{r}'-\mathbf{a}|} - \frac{GM_2}{|\mathbf{r}'-\mathbf{b}|} - \tfrac{1}{2}\omega^2 \mathbf{r}'^2 \tag{7.4.8a}$$

or $V(x', y')$ in Cartesian coordinates

$$V(x', y') = -\frac{GM_1}{\sqrt{(x'-a)^2 + y'^2}} - \frac{GM_2}{\sqrt{(x'+b)^2 + y'^2}} - \tfrac{1}{2}\omega^2 (x'^2 + y'^2) \tag{7.4.8b}$$

The last term in Equations 7.4.7a and b is velocity-dependent and cannot be expressed as the gradient of an effective potential. Thus, we must include the Coriolis term as an additional term in any equation that derives the force from the effective potential. For example, Equation 7.4.3 becomes

$$\mathbf{F}' = -\nabla V(x', y') - 2m\boldsymbol{\omega} \times \mathbf{v} \tag{7.4.9}$$

A considerable simplification in all further calculations may be achieved by expressing mass, length, and time in units that transform $V(x', y')$ into an invariant form that makes it applicable to all restricted three-body situations regardless of the values of their masses. First, we scale all distances to the total separation of the two primaries; that is, we let $a + b$ equal *one length unit.* This is analogous to the convention in which the *astronomical unit*, or AU, the mean distance between the Earth and the Sun, is used to express distances to the other planets in the solar system. Next, we set the factor $G(M_1 + M_2)$, equal to one *"gravitational"* mass unit. The "gravitational" masses GM_i of each body can then be expressed as fractional multiples α_i of this unit. Finally, we set the orbital period of the primaries τ equal to 2π *time units.* This implies that the angular velocity of the two primaries about their center of mass and, by association, the rate of rotation of the x'-y' frame of reference, is $\omega = 1$ *inverse time unit.* Use of these scaled units allows us to characterize the equations of motion by the single parameter α, where $0 < \alpha < 0.5$. In addition, it has the added benefit of riding our expressions of the obnoxious factor G.

In terms of α, the distance of each primary from the center of mass is then

$$\alpha = \frac{a}{a+b} \qquad \beta = \frac{b}{a+b} = 1 - \alpha \tag{7.4.10}$$

The coordinates of the first primary are, thus, $(\alpha, 0)$ and those of the second primary are $(1 - \alpha, 0)$. Furthermore, because the origin of the coordinate system is the center of mass, from Equation 7.3.1, we have

$$M_1 a = M_2 b \tag{7.4.11}$$

and the "gravitational" masses of each primary can then be expressed also in terms of the factor α

$$\alpha_1 = \frac{GM_1}{G(M_1+M_2)} = \frac{b}{a+b} = 1-\alpha \tag{7.4.12a}$$

$$\alpha_2 = \frac{GM_2}{G(M_1+M_2)} = \frac{b}{a+b} = \alpha \tag{7.4.12b}$$

M_1 is the mass of the larger primary, and M_2 is the mass of the smaller one, hence, $0 < \alpha < 0.5$ and $0.5 < 1-\alpha < 1$.

EXAMPLE 7.4.1

Using the previously discussed units, describe the general properties for the binary star system in Example 7.3.1. The mass of the Sun is $M_\odot = 1.99 \times 10^{30}$ kg. The astronomical unit is 1 AU = 1.496×10^{11} m.

Solution:

The masses of the two primaries: M_i	$4\,M_\odot$ and $1\,M_\odot$, respectively
The parameter α:	$1/(1+4) = 0.2$.
The scaled masses of the two primaries α_i:	$1-\alpha = 0.8$; $\alpha = 0.2$.
Coordinates (x_i', y_i') of the two primaries:	(0.2, 0), (−0.8, 0)
Unit of "gravitational" mass $G(M_1+M_2)$:	6.6×10^{20} m^3/s^2
Orbital period: $\tau = 5$ years $= 2\pi$ time units	1.58×10^8 s
Unit of time: $\tau/2\pi$	2.51×10^7 s (0.796 year)
Angular speed: $\omega = 2\pi/\tau$ (=1 inverse time unit)	3.98×10^{-8} s^{-1}
Unit of length: $a+b = 5$ AU	7.48×10^{11} m

In terms of these new units, the effective potential function of Equation 7.4.8b becomes

$$V(x',y') = -\frac{1-\alpha}{\sqrt{(x'-\alpha)^2+y'^2}} - \frac{\alpha}{\sqrt{(x'+1-\alpha)^2+y'^2}} - \frac{x'^2+y'^2}{2} \tag{7.4.13}$$

A plot of the effective potential $V(x', y')$ is shown in Figure 7.4.2 for the Earth–Moon primary system, where the parameter $\alpha = 0.0121$. Plots of the effective potential of other binary systems, such as binary stars where the parameter α is rarely less than 20% or, at the other extreme, the Sun–Jupiter system where $\alpha = 0.000953875$, are qualitatively identical.

It is worth taking the time to examine this plot closely because it exhibits a number of features that give us some insight into the possible orbits of the tertiary.

- $V(x', y') \to -\infty$ at the location of the two primaries. These points are singularities. This is a consequence of the fact that each primary has been treated as though it was a point mass. We might imagine that, if a tertiary were embedded somewhere

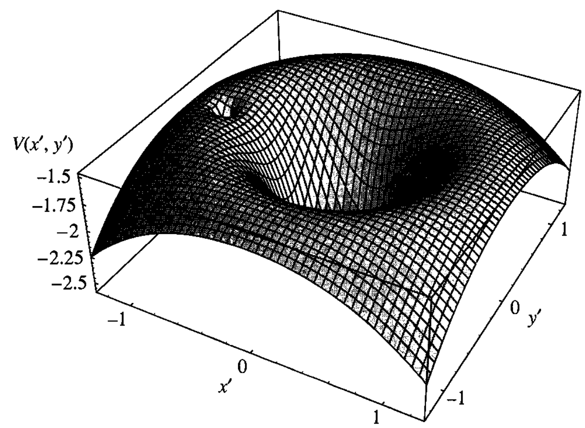

Figure 7.4.2 Effective potential $V(x', y')$ for the Earth–Moon system.

within one of those potential "holes," it might orbit that primary as though the other primary didn't even exist. As an example, consider the Sun–Jupiter system: Each primary is the source of an accouterment of "satellites"; Jupiter has its moons and the Sun has its four inner, terrestrial planets. Neither primary interferes with the attachments of the other (at least not very much). Note, though, that the angular speeds of all these "satellites" about their respective primary are much greater than the angular speed of the two primaries about their center of mass. In addition, tertiaries in such orbits are dragged along by the primary in its own orbit.

- $V(x', y') \to -\infty$ as either x' or $y' \to \infty$. This is a consequence of the rotation of the $x'y'$ coordinate system. In essence, any tertiary initially at rest with respect to this rotating system, but far from the center of mass of the two primaries, experiences a large centrifugal force that tends to move the body even farther from the origin. Eventually, such a body might find itself in a stable orbit at some remote distance $[r' > (a + b)]$ around the center of mass of the two primaries but *not at rest in the rotating frame of reference.* The angular speed of such a tertiary would be so much smaller than the angular speed of the two primaries that a stable, counterclockwise, *prograde* orbit in a fixed frame of reference would appear to be a stable, clockwise, *retrograde* orbit in the rotating system, with an angular velocity that is the negative of that of the primaries. An example of this is our nearest stellar neighbor, the three-body, α-Centauri star system, made up of two primaries, α-Centauri A and B and a tertiary, Proxima Centauri (Figure 7.4.3).
- There are five locations where $\nabla V(x', y') = 0$, or where the force on a particle at rest in the $x'y'$ frame of reference vanishes. These points are called the *Lagrangian points,* after Joseph-Louis Lagrange. They are designated L_1–L_5 in his honor. Three of these points are collinear, lying along the x'-axis. L_1 lies between the two primaries. L_2 lies on the side opposite the least massive primary, and L_3 lies on the side opposite the most massive primary. These three points are saddle points of $V(x', y')$. Along the x' direction they are local maxima, but along the y' direction they are local minima.
- The two primaries form a common base of two equilateral triangles at whose apex lie the points L_4 and L_5, which are absolute maxima of the function $V(x', y')$. As

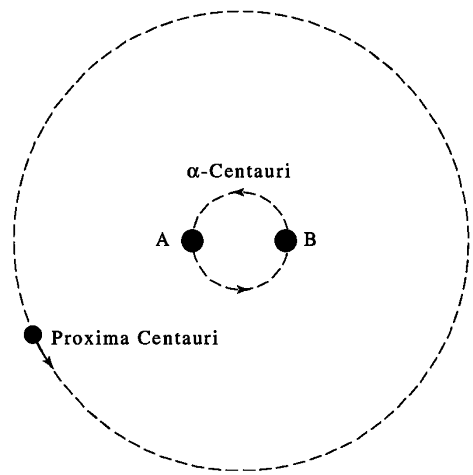

Figure 7.4.3 The α-Centauri system. The masses of the primaries are 1.1 and $0.88M_{\odot}$. The mass of Proxima Centauri is $0.1M_{\odot}$. A and B separated by 25 AU, and Proxima Centauri orbits the pair at a distance of 50,000 AU.

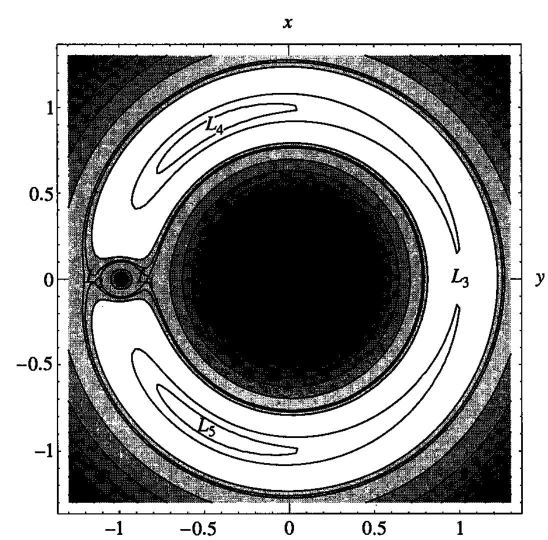

Figure 7.4.4 Contour plot of the effective potential $V(x', y')$ for the Earth–Moon system.

the primaries rotate about their center of mass, L_4 remains 60° ahead of the least massive primary (in the $+y'$ direction), and L_5 remains 60° behind it (in the $-y'$ direction). The location of these five points can be more easily visualized by examining a contour plot of the effective potential function shown in Figure 7.4.4.

- Each line in the contour plot is an equipotential, that is, a line that satisfies the condition $V(x_i', y_i') = V_i$, where V_i is a constant. Normally, the equipotential lines in contour plots represent "heights" V_i that differ from one another by equal amounts. This means that regions of the plot where the gradient, $\nabla V(x', y')$, is "steep" (or the force is large) would exhibit closely packed contour lines. Regions where the gradient is "flat" (or the force approaches zero) would exhibit sparsely packed contour lines. We have not adhered to this convention in Figure 7.4.4. We have decreased the "step size" between contour heights that pass near the five Lagrangian points to illuminate those positions more clearly.

You might guess that it would be possible for a tertiary to remain at any one of these five points, synchronously locked to the two primaries as they rotate about their center of mass. It turns out that this never happens in nature for any tertiary located at L_1–L_3. These are points of unstable equilibrium. If a body located at one of these points is perturbed ever so slightly, it moves toward one primary and away from the other, or away from both primaries.

Close examination of Figures 7.4.3 and 7.4.4 reveals that the effective potential is rather flat and broad around L_4 and L_5, suggesting that a reasonably extensive, almost force-free, region exists where a tertiary might comfortably sit, more or less balanced by the opposing action of the gravitational and centrifugal forces. Because L_4 and L_5 are locations of absolute maxima, however, you might also guess that no stable, synchronous orbit is possible at these points either. Remember, though, that all the forces acting on the tertiary are not derived from the gradient of $V(x', y')$. The velocity-dependent Coriolis force must be considered and it has a nonnegligible effect, particularly in any region where it dominates, which under certain conditions can be the case in the region surrounding L_4 and L_5. The Coriolis force always acts perpendicular to the velocity of a particle. Thus, it does not alter its kinetic energy because $\mathbf{F} \cdot \mathbf{v} = 0$. If a tertiary is nearly stationary in the $x'y'$ frame of reference, moving slowly in the proper direction near either L_4 or L_5, the Coriolis force might dominate the nearly balanced gravitational and centrifugal forces and simply redirect its velocity, causing the tertiary to circulate around L_4 or L_5. In fact, this can and does happen in nature. The Coriolis force creates an effective, quasi-elliptical barrier around the L_4 and L_5 points, thus, turning the maxima of the effective potential into small "wells" of stability. Given the right conditions, we might expect the tertiary to closely follow one of the equipotential contours around L_4 and L_5, both its kinetic and potential energies remaining fairly constant throughout its motion.

The situation just described is analogous to the circulation of air that occurs around high-pressure systems, or "bumps," in the Earth's atmosphere. Gravity tries to pull the air toward the Earth; centrifugal force tries to throw it out; as air spills down from the high, the Coriolis force causes it to circulate about the high-pressure bump, clockwise in the Northern Hemisphere. Such circulating systems in the atmosphere of Earth are only stable temporarily. They form and then dissipate. The Great Red Spot on Jupiter, however, is a high-pressure storm that is a permanent feature of its atmosphere—permanent in the sense that it has been there ever since Galileo saw it with his telescope about 400 years ago! Note that these circulatory patterns are "stationary" with respect to the rotating system. The same holds true for the orbit of a tertiary around L_4 and L_5.

The Trojan Asteroids

The Trojan asteroids are a particular group of asteroids in a 1:1 orbital resonance with Jupiter and whose centroids lie along the orbit of Jupiter, 60° ahead of it and 60° behind (see Figure 7.4.5). These are the L_4 and L_5 points in the Sun–Jupiter primary system. Notice that the Trojans are spread out somewhat diffusely about the L_4 and L_5 points. Each member of the group rotates with Jupiter about the Sun in a fixed frame of reference

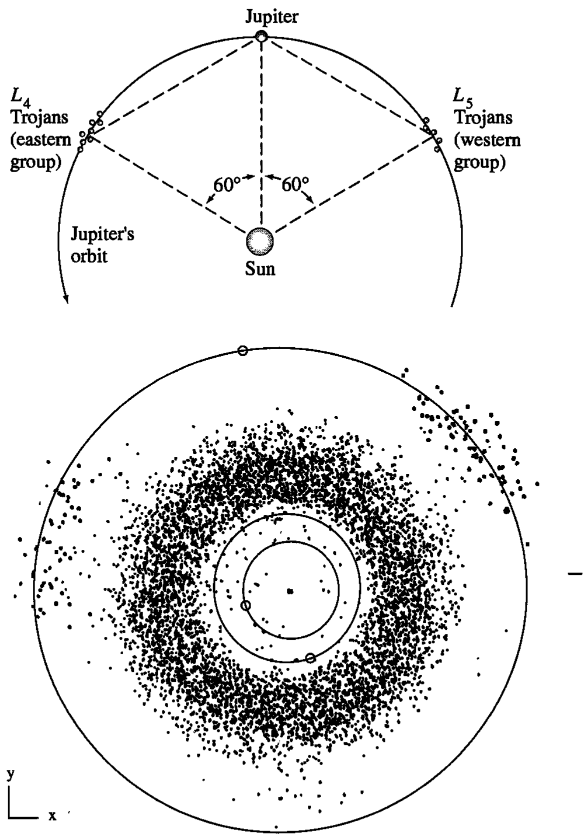

Figure 7.4.5 (a) The Trojan asteroids. (b) Trojan asteroids and the asteroid belt shown with orbits of Jupiter, Mars and Earth.

but is slowly circulating clockwise about L_4 and L_5, as viewed from above in the $x'y'$ frame of reference. In this section, we calculate some examples of the orbits of these asteroids.

First, we rewrite the equations of motion (Equations 7.4.7a and b) using the scaled coordinates we just introduced. Letting

$$r_1' = \sqrt{(x'-\alpha)^2 + y'^2} \qquad r_2' = \sqrt{(x'+1-\alpha)^2 + y'^2} \tag{7.4.14}$$

Equations 7.4.7a and b become

$$\ddot{x}' = -(1-\alpha)\frac{(x'-\alpha)}{r_1'^3} - \alpha\frac{(x'+1-\alpha)}{r_2'^3} + x' + 2\dot{y}' \tag{7.4.15a}$$

$$\ddot{y}' = -(1-\alpha)\frac{y'}{r_1'^3} - \alpha\frac{y'}{r_2'^3} + y' - 2\dot{x}' \tag{7.4.15b}$$

In Example 4.3.2, we employed *Mathematica*'s numerical differential equation solver, *NDSolve*, to solve a set of coupled, second-order differential equations like the ones in Equations 7.4.15a and b. We employ the same technique here with one minor difference: we introduce two additional variables u' and v', such that

$$\dot{x}' = u' \tag{7.4.16a}$$

$$\dot{y}' = v' \tag{7.4.16b}$$

to convert the pair of second-order equations in Equations 7.4.15a and b into two first-order ones

$$\dot{u}' = -(1-\alpha)\frac{(x'-\alpha)}{r_1'^3} - \alpha\frac{(x'+1-\alpha)}{r_2'^3} + x' + 2v' \tag{7.4.16c}$$

$$\dot{v}' = -(1-\alpha)\frac{y'}{r_1'^3} \qquad -\alpha\frac{y'}{r_2'^3} \qquad + y' - 2u' \tag{7.4.16d}$$

This was the same trick we used in Section 3.8, where we solved for the motion of the self-limiting oscillator. The trick is a standard ploy used to convert n second-order differential equations into $2n$ first-order ones, making it possible to use Runge-Kutta techniques to solve the resulting equations. Most numerical differential equation solvers use this technique. *Mathcad* requires that the user input the $2n$ equations in first-order form. This is not a requirement in *Mathematica,* although it is still an option. We use the technique because it is so universally applicable. In the following section, we outline the specific call that we made to *NDSolve*. It is analogous to the one discussed in Example 4.3.2. We dropped the superfluous primes used to label the rotating coordinates because *Mathematica* uses primes in place of dots to denote the process of differentiation, that is, x' means $\dot{x}$. We urge you to remember that the variables x, y, u, and v used in *Mathematica* calls refer to the rotating coordinate system, and the number of primes beside a variable refer to the order of the derivative.

NDSolve [{equations, initial conditions}, $\{u, v, x, y\}$, $\{t, t_{min}, t_{max}\}$]

- {*equations, initial conditions*}
  Insert the four numerical differential equations and initial conditions using the following format

$$\begin{aligned}
&\{x'[t] == u[t],\\
&y'[t] == v[t],\\
&u'[t] == -(1-\alpha)(x[t]-\alpha)/r_1(x[t], y[t])^3 - \alpha(x[t]+1-\alpha)/r_2(x[t], y[t])^3\\
&\qquad\qquad + x[t] + 2v[t],\\
&v'[t] == -(1-\alpha)y[t]/r_1(x[t], y[t])^3\\
&\qquad\qquad - \alpha y[t]/r_2(x[t], y[t])^3 + y[t] - 2\,u[t],\\
&x[0] == x_0,\ y[0] == y_0,\ u[0] == u_0,\ v[0] == v_0\}
\end{aligned}$$

- $\{x, y, u, v\}$
  Insert the four dependent variables whose solutions are desired

$$\{x, y, u, v\}$$

- $\{t, t_{min}, t_{max}\}$
  Insert the independent variable and its range over which the solution is to be evaluated $\{t, 0, t_{max}\}$

TABLE 7.4.1 [illegible]

Parameter	Orbit 1	Orbit 2	Orbit 3	Orbit 4	Orbit 5
x_0	−0.509	−0.524	−0.524	−0.509	−0.532
y_0	0.883	0.909	0.920	0.883	0.920
u_0	0.0259	0.0647	0.0780	−0.0259	0.0780
v_0	0.0149	0.0367	0.0430	−0.049	0.0430
T (units)	80.3	118	210.5	80.3*	—
T (years)	152	223	397	152*	—

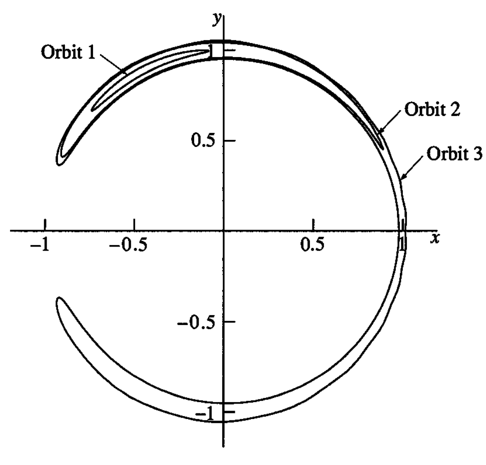

Figure 7.4.6 Orbits 1, 2, 3 of the Trojan asteroids corresponding to the conditions given in Table 7.4.1.

Note, the two functions $r_1[x, y]$ and $r_2[x, y]$ (see Equation 7.5.14) must be defined in *Mathematica* before the call to *NDSolve*. This is also true for the initial conditions x_0, y_0, u_0, and v_0 and the value of α. The value of α for the Sun–Jupiter system is 0.000953875.

We calculated orbits for five sets of initial conditions, in each case starting the tertiary near L_4. The starting conditions and period of the resulting orbit (if the result is a stable orbit) are shown in Table 7.4.1.

As before (Example 4.3.2) we used *Mathematica's ParametricPlot* to generate plots of each of the orbits whose initial conditions are given in Table 7.4.1. Plots of the first three orbits are shown in Figure 7.4.6.

The unit of length is the mean distance between Jupiter and the Sun, $a + b = 5.203$ AU, or about 7.80×10^{11} m. The unit of time was defined such that one rotational period of the primary system, the orbital period of Jupiter ($T_J = 11.86$ years), equals 2π time units. Thus, one time unit equals $T_J/2\pi = 1.888$ years. Tertiaries that follow orbits 1 and 2 circulate slowly, clockwise, around L_4. Their calculated periods are 80.3 and 118 time units, respectively. Using the conversion factor gives us the periods of their orbits in years listed in the last row of Table 7.4.1. Orbit 3 is particularly interesting. The tertiary starts closer to Jupiter than do the other two and moves slowly over L_4 and back around the Sun, more or less along Jupiter's orbital path. It then slowly migrates toward Jupiter, passing under

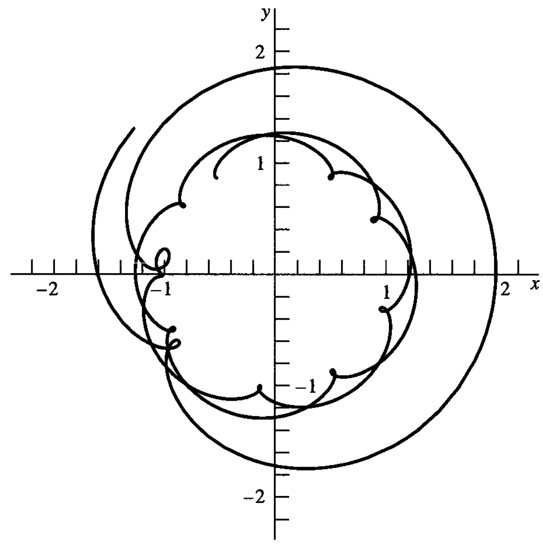

Figure 7.4.7 Trojan asteroids—orbit 4 (see Table 7.4.1).

L_5 and approaching as close to Jupiter as when it started near L_4. Then it loops around L_5, passes back around the Sun, and moves back toward Jupiter, passing just under L_4 to the point where it started. The period of this orbit is 397 years.

Notice that the orbits closely follow the equipotential contours shown in Figure 7.4.4. This is not too surprising because as we remarked earlier, the Coriolis force does not change the kinetic energy of the tertiary. Thus, because the gravitational and centrifugal forces are more or less in balance, the orbits ought to follow the equipotential contours rather closely. The contours circulate around L_4 and L_5 individually, as do orbits 1 and 2, and a few contours circulate around L_4 and L_5 together, as does orbit 3. Given the shape of these orbits, it is easy to understand why the Trojan asteroids appear to be the rather loosely strung out cluster that you see in Figure 7.4.5.

In all cases, the orbits circulate in clockwise fashion like the air around high pressures in the Northern Hemisphere of Earth. The Coriolis force is directed "inward" for clockwise rotation and "outward" for counterclockwise rotation because of the sign of $\boldsymbol{\omega} \times \mathbf{v}$. Orbit 4, shown in Figure 7.4.7, reflects the consequences of a sign reversal in $\boldsymbol{\omega} \times \mathbf{v}$ if we try to set up a counterclockwise circulation about L_4. The orbit was generated with the same parameters as those of the stable orbit 1, except the sign of the initial velocity was reversed. The tertiary, after executing several loopty-loops, is soon thrown completely out of the region between Jupiter and the Sun. A velocity reversal like this would have no effect on the shape of a Keplerian orbit about a single, central gravitational force. The resulting stable orbit would simply be a reversed direction, retrograde orbit. Although most orbits in the solar system are prograde (counterclockwise as seen from above the plane of the ecliptic), retrograde (clockwise) orbits do occur, as, for example, Triton, Neptune's major moon. Reversed orbits are not possible around L_4 and L_5.

The conditions for the stability of these clockwise orbits around L_4 and L_5 have been studied in much more detail than can be presented here. The interested reader is referred

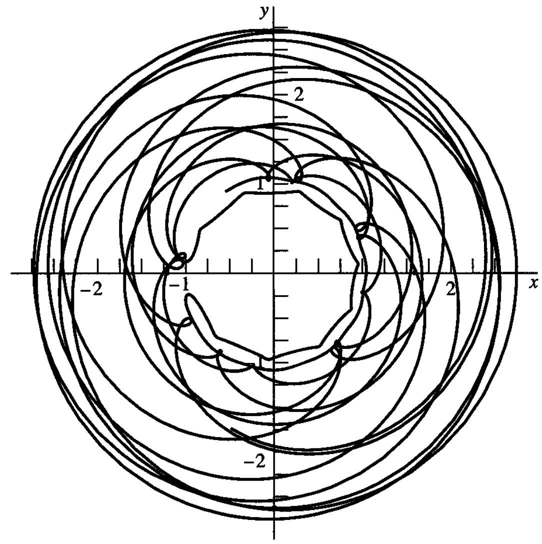

Figure 7.4.8 Trojan asteroids—orbit 5 (see Table 7.4.1).

to the book by V. Szebehely referenced in footnote 4. Stable orbits are only possible for values of the mass parameter $\alpha_c = 0.03852$. The Jupiter–Sun system easily meets that condition, but the orbits of some of the Trojans are only marginally stable. This is particularly true for orbits such as orbit 3. Perturbations, if large enough, can have dramatic consequences for tertiaries in such orbits. Examine the starting conditions for orbit 5, which are virtually identical to those for orbit 3 except for the initial x' coordinate, which was changed by about 2%. The resultant "orbit" is shown in Figure 7.4.8. The trajectory of the tertiary was followed for 300 time units, or about 566 years. Eventually, as was the case for orbit 4, the asteroid was thrown completely out of the L_4–L_5 region, finally settling down in orbit about both primaries at a distance of about 3 units, or 15 AU, which places it somewhere between Saturn and Neptune. In fact, Jupiter is believed to have had just this effect on many of the asteroids that existed near it during the formative stages of the solar system.

Are there any other examples of objects orbiting primaries at the L_4 and L_5 points? A prime example is that of a number of Saturn's large supply of moons. Telesto and Calypso, two moons discovered by the *Voyager* mission, share an orbit with Tethys. Saturn and Tethys are the primaries, and Telesto is at L_4 and Calypso at L_5. Helene and Dione share another orbit that is 1.28 times farther from Saturn than the one occupied by Tethys, Telesto, and Calypso. Helene is located at the L_4 point of this orbit, and Dione is the primary. No moon is found for this orbit at L_5.

A number of space colony enthusiasts have argued that a large space colony could be deployed in a stable orbit at L_5 of the Earth–Moon primary system.[7] The mass parameter for the Earth–Moon system is $\alpha = 0.0121409$, which is certainly less than the critical

[7] G. K. O'Neill, "The Colonization of Space," *Phys. Today*, pp. 32–40 (September, 1974).

value α_c, so one might guess that orbits about L_5 would be stable. The Sun would exert perturbations on such an orbiting colony, however, and it is not obvious that its orbit would remain stable for long. This particular restricted four-body problem was only solved recently, in 1968. Quasi-elliptical orbits around L_5, with excursions limited to a few tenths of the Earth–Moon distance, were found to be stable.[8] If one adds the effects of Jupiter to the problem, however, long-term stability becomes problematical. The industrious student might want to tackle this problem numerically.

EXAMPLE 7.4.2

Calculate the coordinates of the L_1–L_3 collinear Lagrange points for the Earth–Moon system and the values of the effective potential function at those points.

Solution:

These three collinear Lagrange points all lie along the x'-axis, where $y' = 0$. These points represent extrema of the effective potential function, $V(x', y')$. Normally, we would find these points by searching for solutions of the equation

$$\frac{\partial}{\partial x'} V(x', y')\bigg|_{y'=0} = 0$$

Mathematica, however, has a tool, its *FindMinimum* function, that allows us to locate minima of functions directly, without first calculating their derivatives. *Mathematica* saves us a lot of work by effectively taking these derivatives for us. The Lagrange points, L_1–L_3, are located at the *maxima* of $V(x', y' = 0)$, however, so, to use *Mathematica's FindMinimum*, we need to pass to it a function $f(x') = -V(x', y' = 0)$ whose *minima* are the locations of L_1–L_3.

$$f(x') = -V(x', y')\bigg|_{y'=0} = \frac{1-\alpha}{|x'-\alpha|} + \frac{\alpha}{|x'-(\alpha-1)|} + \frac{x'^2}{2}$$

We have written the denominators in the preceding equation as absolute values to emphasize that they are positive definite quantities regardless of the value of x' relative to the critical values α and $\alpha - 1$. When we pass $f(x')$ to *Mathematica's FindMinimum* function, we need to ensure that: (1) *FindMinimum* can calculate the derivatives of $f(x')$ because that is one of the things it does in attempting to locate the minima and that (2) the values in the denominator remain positive definite regardless of any action that *FindMinimum* takes on $f(x')$. Thus, we need to remove the absolute values in the denominators of $f(x')$ to eliminate any possible pathologies in the derivative-taking process, but then we must replace their effect, for example, by multiplying the first two terms in the expression by a "step" function defined to take on the values ± 1 depending on the value of x' relative to α and $\alpha - 1$. We call this "step" function sgn(x) and define it to equal -1 when its argument $x < 0$ and $+1$ when $x > 0$.

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[8] R. Kolenkiewicz, L. Carpenter, "Stable Periodic Orbits About the Sun-Perturbed Earth-Moon Triangular Points," *AIAA J.* 6, 7, 1301 (1968).

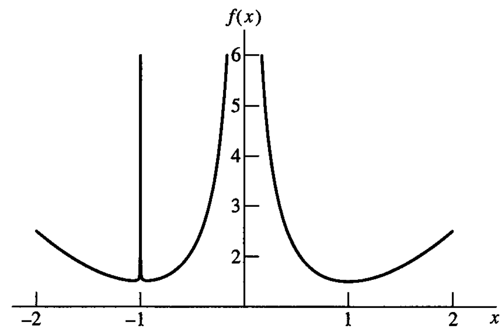

Figure 7.4.9 Regions of applicability for the sgn functions.

TABLE 7.4.2

Call	Lagrange Point	x_0	$\mathrm{sgn}(x - \alpha)$	$\mathrm{sgn}(x - (\alpha - 1))$	x_{min}	$f(x_{min})$
1	L_2	−1.2	−1	−1	−1.06883	1.51874
2	L_1	−0.8	−1	+1	−0.932366	1.51938
3	L_3	1.0	+1	+1	1.0004	1.50048

Inserting it into the expression above gives

$$f(x') = \mathrm{sgn}(x' - \alpha)\frac{1-\alpha}{x'-\alpha} + \mathrm{sgn}(x' - (\alpha-1))\frac{\alpha}{x'-(\alpha-1)} + \frac{x'^2}{2}$$

We can now pass the preceding function to *FindMinimum.* The sgn function takes on a value that insures that the terms in the equation always remain positive regardless of the region along the x'-axis that is being searched for one of the minima of $f(x')$. We also need to pass *FindMinimum* initial values of x' to begin the search. We plot $f(x')$ in Figure 7.4.9 to find approximate locations of the three minima we are using as these starting points. L_2 is the minimum located exterior to the singularity at $x' = -1$ that represents the location of Jupiter. Thus, $L_2 \approx -(1 + \epsilon)$. L_1 is located on the interior side of this singularity. Thus, $L_1 \approx -(1 - \epsilon)$ and L_3 is located just beyond the mirror image of Jupiter's singularity at $x' = +1$ opposite the Sun. Thus, $L_3 \approx +(1 + \epsilon)$. ϵ simply denotes some unknown small value. We now make three calls to *FindMinimum* to locate each of the three collinear Lagrange points.

Each call takes the form: *Find Minimum* [*function*, {x, x_0}] where the argument *function* means $f(x)$ as previously defined. Again, we drop the prime notation. x is the independent variable of the function, and x_0 is the value used to start the search. Table 7.4.2 lists the parameters input to each call. The output of the call are the locations x_{min} of the Lagrange points and the corresponding values of $f(x_{min})$. The values of x_0 were chosen to ensure that the search starts in the region in which the desired Lagrange point is located and fairly near to it.

7.5 Collisions

Whenever two bodies undergo a collision, the force that either exerts on the other during the contact is an internal force, if the bodies are regarded together as a single system. The total linear momentum is unchanged. We can, therefore, write

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2' \tag{7.5.1a}$$

or, equivalently,

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2' \tag{7.5.1b}$$

The subscripts 1 and 2 refer to the two bodies, and the primes indicate the respective momenta and velocities after the collision. Equations 7.5.1a and b are quite general. They apply to any two bodies regardless of their shapes, rigidity, and so on.

With regard to the energy balance, we can write

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2} + Q \tag{7.5.2a}$$

or

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}m_1v_1'^2 + \tfrac{1}{2}m_2v_2'^2 + Q \tag{7.5.2b}$$

Here the quantity Q is introduced to indicate the net loss or gain in kinetic energy that occurs as a result of the collision.

In the case of an *elastic* collision, no change takes place in the total kinetic energy, so that $Q = 0$. If an energy loss does occur, then Q is positive. This is called an *exoergic* collision. It may happen that an energy gain occurs. This would happen, for example, if an explosive was present on one of the bodies at the point of contact. In this case Q is negative, and the collision is called *endoergic.*

The study of collisions is of particular importance in atomic, nuclear, and high-energy physics. Here the bodies involved may be atoms, nuclei, or various elementary particles, such as electrons and quarks.

Direct Collisions

Let us consider the special case of a head-on collision of two bodies, or particles, in which the motion takes place entirely on a single straight line, the x-axis, as shown in Figure 7.5.1. In this case the momentum balance equation (Equation 7.5.1b) can be written

$$m_1\dot{x}_1 + m_2\dot{x}_2 = m_1\dot{x}_1' + m_2\dot{x}_2' \tag{7.5.3}$$

The direction along the line of motion is given by the signs of the $\dot{x}$'s.

To compute the values of the velocities after the collision, given the values before the collision, we can use the preceding momentum equation together with the energy balance equation (Equation 7.5.2b), if we know the value of Q. It is often convenient in this kind of problem to introduce another parameter ϵ called the *coefficient of restitution.*

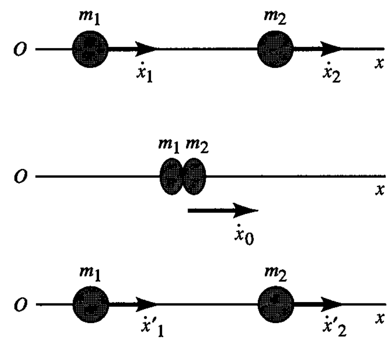

Figure 7.5.1 Head-on collision of two particles.

This quantity is defined as the ratio of the speed of separation v' to the speed of approach v. In our notation ϵ may be written as

$$\epsilon = \frac{|\dot{x}_2' - \dot{x}_1'|}{|\dot{x}_2 - \dot{x}_1|} = \frac{v'}{v} \tag{7.5.4}$$

The numerical value of ϵ depends primarily on the composition and physical makeup of the two bodies. It is easy to verify that in an elastic collision the value of $\epsilon = 1$. To do this, we set $Q = 0$ in Equation 7.5.2b and solve it together with Equation 7.5.3 for the final velocities. The steps are left as an exercise.

In the case of a *totally inelastic* collision, the two bodies stick together after colliding, so that $\epsilon = 0$. For most real bodies ϵ has a value somewhere between the two extremes of 0 and 1. For ivory billiard balls it is about 0.95. The value of the coefficient of restitution may also depend on the speed of approach. This is particularly evident in the case of a silicone compound known as Silly Putty. A ball of this material bounces when it strikes a hard surface at high speed, but at low speeds it acts like ordinary putty.

We can calculate the values of the final velocities from Equation 7.5.3 together with the definition of the coefficient of restitution (Equation 7.5.4). The result is

$$\begin{aligned} \dot{x}_1' &= \frac{(m_1 - \epsilon m_2)\dot{x}_1 + (m_2 + \epsilon m_2)\dot{x}_2}{m_1 + m_2} \\ \dot{x}_2' &= \frac{(m_1 + \epsilon m_1)\dot{x}_1 + (m_2 - \epsilon m_1)\dot{x}_2}{m_1 + m_2} \end{aligned} \tag{7.5.5}$$

Taking the totally inelastic case by setting $\epsilon = 0$, we find, as we should, that $\dot{x}_1' = \dot{x}_2'$; that is, there is no rebound. On the other hand, in the special case that the bodies are of equal mass $m_1 = m_2$ and are perfectly elastic $\epsilon = 1$, we obtain

$$\begin{aligned} \dot{x}_1' &= \dot{x}_2 \\ \dot{x}_2' &= \dot{x}_1 \end{aligned} \tag{7.5.6}$$

The two bodies, therefore, just exchange their velocities as a result of the collision.

In the general case of a direct nonelastic collision, it is easily verified that the energy loss Q is related to the coefficient of restitution by the equation

$$Q = \tfrac{1}{2}\mu v^2(1 - \epsilon^2) \tag{7.5.7}$$

in which $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass, and $v = |\dot{x}_2 - \dot{x}_1|$ is the relative speed before impact. The derivation is left as an exercise (see Problem 7.9).

Impulse in Collisions

Forces of extremely short duration in time, such as those exerted by bodies undergoing collisions, are called *impulsive forces*. If we confine our attention to one body, or particle, the differential equation of motion is $d(m\mathbf{v})/dt = \mathbf{F}$, or in differential form $d(m\mathbf{v}) = \mathbf{F}\,dt$. Let us take the time integral over the interval $t = t_1$ to $t = t_2$. This is the time during which the force is considered to act. Then we have

$$\Delta(m\mathbf{v}) = \int_{t_1}^{t_2} \mathbf{F}\,dt \tag{7.5.8a}$$

The time integral of the force is the impulse. It is customarily denoted by the symbol $\boldsymbol{P}$. Equation 7.5.8a is, accordingly, expressed as

$$\Delta(m\mathbf{v}) = \boldsymbol{P} \tag{7.5.8b}$$

We can think of an *ideal impulse* as produced by a force that tends to infinity but lasts for a time interval that approaches zero in such a way that the integral $\int \mathbf{F}\,dt$ remains finite. Such an ideal impulse would produce an instantaneous change in the momentum and velocity of a body without producing any displacement.

EXAMPLE 7.5.1

Determining the Speed of a Bullet

A gun is fired horizontally, point-blank at a block of wood, which is initially at rest on a horizontal floor. The bullet becomes imbedded in the block, and the impact causes the system to slide a certain distance s before coming to rest. Given the mass of the bullet m, the mass of the block M, and the coefficient of sliding friction between the block and the floor μ_k, find the initial speed (muzzle velocity) of the bullet.

Solution:

First, from conservation of linear momentum, we can write

$$m\dot{x}_0 = (M + m)\dot{x}_0'$$

where $\dot{x}_0$ is the initial velocity of the bullet, and $\dot{x}_0'$ is the velocity of the system (block + bullet) immediately after impact. (The coefficient of restitution ϵ is zero in this case.) Second, we know that the magnitude of the retarding frictional force is equal to $(M + m)\,\mu_k g = (M + m)a$, where $a = -\ddot{x}$ is the deceleration of the system after impact, so $a = \mu_k g$. Now, from Chapter 2 we recall that $s = v_0^2/2a$ for the case of uniform acceleration in one dimension. Thus, in our problem

$$s = \frac{\dot{x}_0'^2}{2\mu_k g} = \left(\frac{m\dot{x}_0}{M + m}\right)^2 \left(\frac{1}{2\mu_k g}\right)$$

Solving for $\dot{x}_0$, we obtain

$$\dot{x}_0 = \left(\frac{M+m}{m}\right)(2\mu_k g s)^{1/2}$$

for the initial velocity of the bullet in terms of the given quantities.

As a numerical example, let the mass of the block be 4 kg, and that of the bullet 10 g = 0.01 kg (about that of a .38 calibre slug). For the coefficient of friction (wood-on-wood) let us take $\mu_k = 0.4$. If the block slides a distance of 15 cm = 0.15 m, then we find

$$\dot{x}_0 = \frac{4.01}{0.01}(2 \times 0.4 \times 9.8 \text{ ms}^{-2} \times 0.15 \text{ m})^{1/2} = 435 \text{ m/s}$$

7.6 Oblique Collisions and Scattering: Comparison of Laboratory and Center of Mass Coordinates

We now turn our attention to the more general case of collisions in which the motion is not confined to a single straight line. Here the vectorial form of the momentum equations must be employed. Let us study the special case of a particle of mass m_1 with initial velocity $\mathbf{v}_1$ (the incident particle) that strikes a particle of mass m_2 that is initially at rest (the target particle). This is a typical problem found in nuclear physics. The momentum equations in this case are

$$\mathbf{p}_1 = \mathbf{p}_1' + \mathbf{p}_2' \tag{7.6.1a}$$

$$m_1\mathbf{v}_1 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2' \tag{7.6.1b}$$

The energy balance condition is

$$\frac{p_1^2}{2m_1} = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2} + Q \tag{7.6.2a}$$

or

$$\tfrac{1}{2}m_1 v_1^2 = \tfrac{1}{2}m_1 v_1'^2 + \tfrac{1}{2}m_2 v_2'^2 + Q \tag{7.6.2b}$$

Here, as before, the primes indicate the velocities and momenta after the collision, and Q represents the net energy that is lost or gained as a result of the impact. The quantity Q is of fundamental importance in atomic and nuclear physics, because it represents the energy released or absorbed in atomic and nuclear collisions. In many cases the target particle is broken up or changed by the collision. In such cases the particles that leave the collision are different from those that enter. This is easily taken into account by assigning different masses, say m_3 and m_4, to the particles leaving the collision. In any case, the law of conservation of linear momentum is always valid.

Consider the particular case in which the masses of the incident and target particles are the same. Then the energy balance equation (Equation 7.6.2a) can be written

$$p_1^2 = p_1'^2 + p_2'^2 + 2mQ \tag{7.6.3}$$

where $m = m_1 = m_2$. Now if we take the dot product of each side of the momentum equation (Equation 7.6.1a) with itself, we get

$$p_1^2 = (\mathbf{p}_1' + \mathbf{p}_2')\cdot(\mathbf{p}_1' + \mathbf{p}_2') = p_1'^2 + p_2'^2 + 2\mathbf{p}_1' \cdot \mathbf{p}_2' \tag{7.6.4}$$

Comparing Equations 7.6.3 and 7.6.4, we see that

$$\mathbf{p}_1' \cdot \mathbf{p}_2' = mQ \tag{7.6.5}$$

For an elastic collision ($Q = 0$) we have, therefore,

$$\mathbf{p}_1' \cdot \mathbf{p}_2' = 0 \tag{7.6.6}$$

so the two particles emerge from the collision at right angles to each other.

Center of Mass Coordinates

Theoretical calculations in nuclear physics are often done in terms of quantities referred to a coordinate system in which the center of mass of the colliding particles is at rest. On the other hand, the experimental observations on scattering of particles are carried out in terms of the laboratory coordinates. We, therefore, consider briefly the problem of conversion from one coordinate system to the other.

The velocity vectors in the laboratory system and in the center of mass system are illustrated diagrammatically in Figure 7.6.1. In the figure ϕ_1 is the angle of deflection of the incident particle after it strikes the target particle, and ϕ_2 is the angle that the line of motion of the target particle makes with the line of motion of the incident particle. Both ϕ_1 and ϕ_2 are measured in the laboratory system. In the center of mass system, because the center of mass must lie on the line joining the two particles at all times, both particles approach the center of mass, collide, and recede from the center of mass in opposite directions. The angle θ denotes the angle deflection of the incident particle in the center of mass system as indicated.

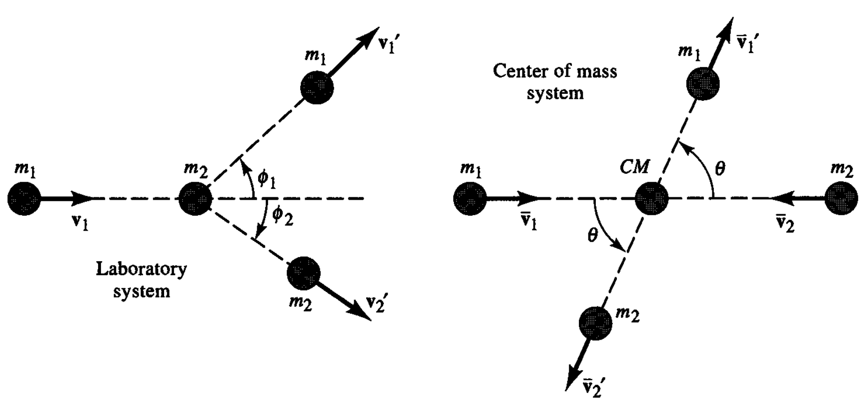

Figure 7.6.1 Comparison of laboratory and center of mass coordinates.

From the definition of the center of mass, the linear momentum in the center of mass system is zero both before and after the collision. Hence, we can write

$$\bar{\mathbf{p}}_1 + \bar{\mathbf{p}}_2 = 0 \tag{7.6.7a}$$

$$\bar{\mathbf{p}}_1' + \bar{\mathbf{p}}_2' = 0 \tag{7.6.7b}$$

The bars are used to indicate that the quantity in question is referred to the center of mass system. The energy balance equation reads

$$\frac{\bar{p}_1^2}{2m_1} + \frac{\bar{p}_2^2}{2m_2} = \frac{\bar{p}_1'^2}{2m_1} + \frac{\bar{p}_2'^2}{2m_2} + Q \tag{7.6.8}$$

We can eliminate $\bar{p}_2$ and $\bar{p}_2'$ from Equation 7.6.8 by using the momentum relations in Equations 7.6.7a and b. The result, which is conveniently expressed in terms of the reduced mass, is

$$\frac{\bar{p}_1^2}{2\mu} = \frac{\bar{p}_1'^2}{2\mu} + Q \tag{7.6.9}$$

The momentum relations, Equations 7.6.7a and b expressed in terms of velocities, read

$$m_1\bar{\mathbf{v}}_1 + m_2\bar{\mathbf{v}}_2 = 0 \tag{7.6.10a}$$

$$m_1\bar{\mathbf{v}}_1' + m_2\bar{\mathbf{v}}_2' = 0 \tag{7.6.10b}$$

The velocity of the center of mass is (see Equations 7.1.3 and 7.1.4)

$$\mathbf{v}_{cm} = \frac{m_1\mathbf{v}_1}{m_1 + m_2} \tag{7.6.11}$$

Hence, we have

$$\bar{\mathbf{v}}_1 = \mathbf{v}_1 - \mathbf{v}_{cm} = \frac{m_2\mathbf{v}_1}{m_1 + m_2} \tag{7.6.12}$$

The relationships among the velocity vectors $\mathbf{v}_{cm}$, $\mathbf{v}_1'$, and $\bar{\mathbf{v}}_1'$ are shown in Figure 7.6.2. From the figure, we see that

$$\begin{aligned} v_1' \sin\phi_1 &= \bar{v}_1' \sin\theta \\ v_1' \cos\phi_1 &= \bar{v}_1' \cos\theta + v_{cm} \end{aligned} \tag{7.6.13}$$

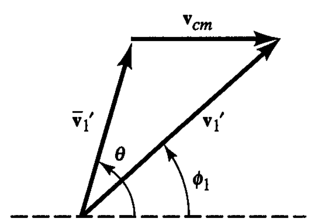

Figure 7.6.2 Velocity vectors in the laboratory system and the center of mass system.

Hence, by dividing, we find the equation connecting the scattering angles to be expressible in the form

$$\tan\phi_1 = \frac{\sin\theta}{\gamma + \cos\theta} \tag{7.6.14}$$

in which γ is a numerical parameter whose value is given by

$$\gamma = \frac{v_{cm}}{\bar{v}_1'} = \frac{m_1 v_1}{\bar{v}_1'(m_1 + m_2)} \tag{7.6.15}$$

The last step follows from Equation 7.6.11.

Now we can readily calculate the value of $\bar{v}_1'$ in terms of the initial energy of the incident particle from the energy equation (Equation 7.6.9). This gives us the necessary information to find γ and, thus, determine the relationship between the scattering angles. For example in the case of an elastic collision $Q = 0$, we find from the energy equation that $\bar{p}_1 = \bar{p}_1'$, or $\bar{v}_1 = \bar{v}_1'$. This result, together with Equation 7.6.12, yields the value

$$\gamma = \frac{m_1}{m_2} \tag{7.6.16}$$

for an elastic collision.

Two special cases of such elastic collisions are instructive to consider. First, if the mass m_2 of the target particle is very much greater than the mass m_1 of the incident particle, then γ is very small. Hence, $\tan\phi_1 \approx \tan\theta$, or $\phi_1 \approx \theta$. That is, the scattering angles as seen in the laboratory and in the center of mass systems are nearly equal.

The second special case is that of equal masses of the incident and target particles $m_1 = m_2$. In this case $\gamma = 1$, and the scattering relation reduces to

$$\begin{aligned} \tan\phi_1 &= \frac{\sin\theta}{1 + \cos\theta} = \tan\frac{\theta}{2} \\ \phi_1 &= \frac{\theta}{2} \end{aligned} \tag{7.6.17}$$

That is, the angle of deflection in the laboratory system is just half that in the center of mass system. Furthermore, because the angle of deflection of the target particle is $\pi - \theta$ in the center of mass system, as shown in Figure 7.6.1, then the same angle in the laboratory system is $(\pi - \theta)/2$. Therefore, the two particles leave the point of impact at right angles to each other as seen in the laboratory system, in agreement with Equation 7.6.6.

In the general case of nonelastic collisions, it is left as a problem to show that γ is expressible as

$$\gamma = \frac{m_1}{m_2}\left[1 - \frac{Q}{T}\left(1 + \frac{m_1}{m_2}\right)\right]^{-1/2} \tag{7.6.18}$$

in which T is the kinetic energy of the incident particle as measured in the laboratory system.

EXAMPLE 7.6.1

In a nuclear scattering experiment a beam of 4-MeV alpha particles (helium nuclei) strikes a target consisting of helium gas, so that the incident and the target particles have equal mass. If a certain incident alpha particle is scattered through an angle of 30° in the laboratory system, find its kinetic energy and the kinetic energy of recoil of the target particle, as a fraction of the initial kinetic energy T of the incident alpha particle. (Assume that the target particle is at rest and that the collision is elastic.)

Solution:

For elastic collisions with particles of equal mass, we know from Equation 7.6.6 that $\phi_1 + \phi_2 = 90°$ (see Figure 7.6.1). Hence, if we take components parallel to and perpendicular to the momentum of the incident particle, the momentum balance equation (Equation 7.6.1a) becomes

$$\begin{aligned} p_1 &= p_1' \cos\phi_1 + p_2' \sin\phi_1 \\ 0 &= p_1' \sin\phi_1 - p_2' \cos\phi_1 \end{aligned}$$

in which $\phi_1 = 30°$. Solving the preceding pair of equations for the primed components, we find

$$\begin{aligned} p_1' &= p_1 \cos\phi_1 = p_1 \cos 30° = \frac{\sqrt{3}}{2} p_1 \\ p_2' &= p_1 \sin\phi_1 = p_1 \sin 30° = \tfrac{1}{2} p_1 \end{aligned}$$

Therefore, the kinetic energies after impact are

$$\begin{aligned} T_1' &= \frac{p_1'^2}{2m_1} = \frac{3}{4}\frac{p_1^2}{2m_1} = \tfrac{3}{4} T = 3 \text{ MeV} \\ T_2' &= \frac{p_2'^2}{2m_2} = \frac{1}{4}\frac{p_1^2}{2m_1} = \tfrac{1}{4} T = 1 \text{ MeV} \end{aligned}$$

EXAMPLE 7.6.2

What is the scattering angle in the center of mass system for Example 7.6.1?

Solution:

Here Equation 7.6.17 gives the answer directly, namely,

$$\theta = 2\phi_1 = 60°$$

EXAMPLE 7.6.3

(a) Show that, for the general case of elastic scattering of a beam of particles of mass m_1 off a stationary target of particles whose mass is m_2, the opening angle ψ in the lab is given by the expression

$$\psi = \phi_1 + \phi_2 = \frac{\pi}{2} + \frac{\phi_1}{2} - \tfrac{1}{2}\sin^{-1}\left(\frac{m_1}{m_2}\sin\phi_1\right)$$

(b) Suppose the beam of particles consists of protons and the target consists of helium nuclei. Calculate the opening angle for a proton scattered elastically at a lab angle $\phi_1 = 30°$.

Solution:

(a) Because particle 2 is at rest in the lab, its center of mass velocity $\bar{v}_2$ is equal in magnitude (and opposite in direction) to v_{cm}. For elastic collisions in the center of mass, momentum and energy conservation can be written as

$$\bar{\mathbf{p}}_1 + \bar{\mathbf{p}}_2 = \bar{\mathbf{p}}_1' + \bar{\mathbf{p}}_2' = 0$$
$$\frac{\bar{p}_1^2}{2m_1} + \frac{\bar{p}_2^2}{2m_2} = \frac{\bar{p}_1'^2}{2m_1} + \frac{\bar{p}_2'^2}{2m_2}$$

Solving for the magnitudes of the center of mass momenta of particle 1 in terms of particle 2, we obtain

$$\bar{p}_1 = \bar{p}_2 \qquad \bar{p}_1' = \bar{p}_2'$$

These expressions can be inserted into the energy conservation equation to obtain

$$\frac{\bar{p}_2^2}{2\mu} = \frac{\bar{p}_2'^2}{2\mu} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$
$$\therefore \bar{v}_2' = \bar{v}_2 = v_{cm}$$

Thus, in an elastic collision, the center of mass velocities of particle 2 are the same before and after the collision, and both are equal to the center of mass velocity. Moreover, the values of the center of mass velocities of particle 1 are also the same before and after the collision, and, from conservation of momentum in the center of mass, they are

$$\bar{v}_1' = \bar{v}_1 = \frac{m_2}{m_1}\bar{v}_2' = \frac{m_2}{m_1}v_{cm}$$

Shown below in Figure 7.6.3 is a vector diagram that relates the parameters of elastic scattering in the laboratory and center of mass frames of reference. From the geometry of Figure 7.6.3, we see that

$$\psi = \phi_1 + \phi_2$$
$$2\phi_2 = \pi - \theta$$
$$\phi_2 = \frac{\pi}{2} - \frac{\theta}{2}$$

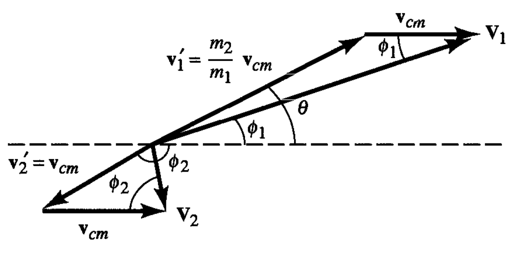

Figure 7.6.3 Velocity vectors in laboratory and center of mass frame for elastic scattering.

Now, applying the law of sines to the upper triangle of the figure, we obtain

$$\frac{(m_2/m_1)v_{cm}}{\sin\phi_1} = \frac{v_{cm}}{\sin(\theta-\phi_1)}$$

$$\sin(\theta-\phi_1) = \frac{m_1}{m_2}\sin\phi_1$$

$$\therefore \theta = \phi_1 + \sin^{-1}\left(\frac{m_1}{m_2}\sin\phi_1\right)$$

Finally, substituting this last expression for θ into the one preceding it for ϕ_2 and solving for the opening angle ψ, we obtain

$$\psi = \phi_1 + \phi_2 = \phi_1 + \left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$= \frac{\pi}{2} + \frac{\phi_1}{2} - \frac{1}{2}\sin^{-1}\left(\frac{m_1}{m_2}\sin\phi_1\right)$$

(b) For elastic scattering of protons off helium nuclei at $\phi_1 = 30°$, $m_1/m_2 = \frac{1}{4}$, and $\psi \approx 101°$.

(Note: In the case where $m_1 = m_2$, $\psi = 90°$ as derived in the text.)

7.7 | Motion of a Body with Variable Mass: Rocket Motion

Thus far, we have discussed only situations in which the masses of the objects under consideration remain constant during motion. In many situations this is not true. Raindrops falling though the atmosphere gather up smaller droplets as they fall, which increases their mass. Rockets propel themselves by burning fuel explosively and ejecting the resultant gasses at high exhaust velocities. Thus, they lose mass as they accelerate. In each case, mass is continually being added to or removed from the body in question, and this change in mass affects its motion. Here we derive the general differential equation that describes the motion of such objects.

So as not to get too confused with signs, we derive the equation by considering the case in which mass is added to the body as it moves. The equation of motion also applies to rockets, but in that case the rate of change of mass is a negative quantity. Examine Figure 7.7.1. A large mass is moving through some medium that is infested with small particles that stick to the mass as it strikes them. Thus, the larger body is continually gathering up mass as it moves through the medium. At some time t, its mass is $m(t)$ and its

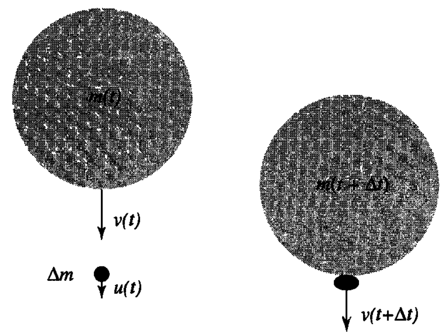

Figure 7.7.1 A mass m gathering up a small mass Δm as it moves through a medium.

velocity is $\mathbf{v}(t)$. The small particles are, in general, not at rest but are moving through the medium also with a velocity that we assume to be $\mathbf{u}(t)$. At time $t + \Delta t$, the large moving object has collided with some of these smaller particles and accumulated an additional small amount of mass Δm. Thus, its mass is now $m(t + \Delta t) = m(t) + \Delta m$ and its velocity has changed to $\mathbf{v}(t + \Delta t)$. In the small time interval Δt, the change (if any) in the total linear momentum of the system is

$$\Delta \mathbf{P} = (\mathbf{p}_{total})_{t+\Delta t} - (\mathbf{p}_{total})_t \tag{7.7.1}$$

This change can be expressed in terms of the masses and velocities before and after the collision

$$\Delta \mathbf{P} = (m + \Delta m)(\mathbf{v} + \Delta \mathbf{v}) - (m\mathbf{v} + \mathbf{u}\,\Delta m) \tag{7.7.2}$$

Because the velocity of Δm relative to m is $\mathbf{V} = \mathbf{u} - \mathbf{v}$, Equation 7.7.2 can be expressed as

$$\Delta \mathbf{P} = m\,\Delta \mathbf{v} + \Delta m\,\Delta \mathbf{v} - \mathbf{V}\,\Delta m \tag{7.7.3}$$

and on dividing by Δt we obtain

$$\frac{\Delta \mathbf{P}}{\Delta t} = (m + \Delta m)\frac{\Delta \mathbf{v}}{\Delta t} - \mathbf{V}\frac{\Delta m}{\Delta t} \tag{7.7.4}$$

In the limit as $\Delta t \to 0$, we have

$$\mathbf{F}_{ext} = \dot{\mathbf{P}} = m\dot{\mathbf{v}} - \mathbf{V}\dot{m} \tag{7.7.5}$$

The force $\mathbf{F}_{ext}$ represents any external force, such as gravity, air resistance, and so forth that acts on the system in addition to the impulsive force that results from the interaction between the masses m and Δm. If $\mathbf{F}_{ext} = 0$, then the total momentum $\mathbf{P}$ of the system is a constant of the motion and its net change is zero. This is the case for a rocket in deep space, beyond the gravitational influence of any planet or star, where $\mathbf{F}_{ext}$ is essentially zero.

We now apply this equation of motion to two special cases in which mass is added to or lost from the moving body. First, suppose that, as we have described, the body is falling through a fog or mist so that it collects mass as it goes, but assume that the small droplets of matter are suspended in the atmosphere such that their initial velocity prior to accretion is zero. In general, this will be a good approximation. Hence, $\mathbf{V} = -\mathbf{v}$, and we obtain

$$\mathbf{F}_{ext} = m\dot{\mathbf{v}} + \mathbf{v}\dot{m} = \frac{d}{dt}(m\mathbf{v}) \tag{7.7.6}$$

for the equation of motion. It applies only if the initial velocity of the matter that is being swept us is zero. Otherwise, the more general Equation 7.7.5, must be used.

For the second case, consider the motion of a rocket. The sign of $\dot{m}$ is negative because the rocket is losing mass in the form of ejected fuel. The term $\mathbf{V}\dot{m}$ in Equation 7.7.5 is called the thrust of the rocket, and its direction is opposite the direction of $\mathbf{V}$, the relative velocity of the exhaust products. Here, we solve the equation of motion for the simplest case of rocket motion in which the external force on it is zero; that is, the rocket is not subject to any force of gravity, air resistance, and so on. Thus, in Equation 7.7.5, $\mathbf{F}_{ext} = 0$, and we have

$$m\dot{\mathbf{v}} = \mathbf{V}\dot{m} \tag{7.7.7}$$

We can now separate the variables and integrate to find $\mathbf{v}$ as follows:

$$\int d\mathbf{v} = \int \frac{\mathbf{V}\,dm}{m} \tag{7.7.8}$$

If we assume that $\mathbf{V}$ is constant, then we can integrate between limits to find the *speed* as function of m:

$$\begin{gathered} \int_{v_0}^{v} dv = -V\int_{m_0}^{m} \frac{dm}{m} \\ v = v_0 + V\ln\frac{m_0}{m} \end{gathered} \tag{7.7.9}$$

Here m_0 is the initial mass of the rocket plus unburned fuel, m is the mass at any time, and V is the speed of the ejected fuel relative to the rocket. Owing to the nature of the logarithmic function, the rocket must have a large fuel-to-payload ratio to attain the large speeds needed for launching satellites into space.

EXAMPLE 7.7.1

Launching an Earth Satellite from Cape Canaveral

We know from Example 6.5.3 that the speed of a satellite in a circular orbit near Earth is about 8 km/s. Satellites are launched toward the east to take advantage of Earth's rotation. For a point on the Earth near the equator the rotational speed is approximately $R_{Earth}\,\omega_{Earth}$, which is about 0.5 km/s. For most rocket fuels the effective ejection speed is of the order of 2 to 4 km/s. For example, if we take $V = 3$ km/s, then we find that the mass ratio calculated from Equation 7.7.9 is

$$\frac{m_0}{m} = \exp\left(\frac{v - v_0}{V}\right) = \exp\left(\frac{8.0 - 0.5}{3}\right) = e^{2.5} = 12.2$$

to achieve orbital speed from the ground. Thus, only about 8% of the total initial mass m_0 is payload.

Multi-Stage Rockets

Example 7.7.1 demonstrates that a large amount of fuel is necessary to put a small payload into low earth orbit (LEO) even if the effects of gravity and air resistance are absent. Neglecting air resistance is not a bad approximation because careful shaping of the rocket can greatly minimize its effect. However, as you most assuredly would suspect, we cannot ignore the effect of gravity because it greatly magnifies the problem of putting something into orbit.

The equation of motion of the rocket with gravity acting is given by Equation 7.7.5

$$m\frac{d\mathbf{v}}{dt} - \mathbf{V}\frac{dm}{dt} = m\mathbf{g} \tag{7.7.10}$$

Choosing the upward direction as positive and rearranging terms, we get

$$\frac{dv}{V} = -\frac{dm}{m} - \frac{g}{V}dt \tag{7.7.11}$$

For the rocket to achieve *liftoff*, the first term on the right of Equation 7.7.11 must exceed the second (remember, dm is negative); in other words, the rocket must eject a lot of matter, dm, at high exhaust velocity V. The reciprocal of the constant g/V in the second term is a "parameter of goodness" for a given type of rocket and has been given a special name, the *specific impulse* τ_s of the rocket engine.

$$\tau_s = \frac{V}{g} \tag{7.7.12}$$

It has the dimensions of time, and its value depends on the exhaust velocity of the rocket. This, in turn, depends primarily on the thermodynamics of what goes on inside the rocket's combustion chamber and the shape of the rocket nozzle. A well-designed chemical rocket that works by rapid oxidation of a fuel typically has an exhaust velocity of about 3000 m/s where the average molecular weight of the combustibles is about 30. Thus, $\tau_s = V/g \approx 300$ s.

We now integrate Equation 7.7.11 during the fuel burn up to the *time of burnout* τ_B to find the final velocity attained by the rocket.

$$\frac{1}{V}\int_0^{v_f} dv = -\int_{m_0}^{m_f}\frac{dm}{m} - \frac{1}{\tau_s}\int_0^{\tau_B} dt \tag{7.7.13a}$$

Completing the integration, we get

$$\frac{v_f}{V} = \ln\left[\frac{m_R + m_p + m_F}{m_R + m_p}\right] - \frac{\tau_B}{\tau_s} \tag{7.7.13b}$$

The masses in the above equation are m_R = mass of the rocket, m_p = mass of the payload, and m_F = mass of the fuel (plus oxidizer).

Solving Equation 7.7.13b for the mass ratio, we get

$$\left[\frac{m_R + m_p + m_F}{m_R + m_p}\right] = e^{\left(\frac{\tau_B}{\tau_s} + \frac{v_f}{V}\right)} \tag{7.7.14}$$

The question of interest here is how much fuel is needed to boost the rocket and payload into LEO? The final velocity of the rocket must be about 8 km/s. Solving for the mass of the fuel relative to the mass of the rocket and its payload, we get

$$\frac{m_F}{m_R + m_p} = e^{\left(\frac{v_f}{V} + \frac{\tau_B}{\tau_s}\right)} - 1 \tag{7.7.15}$$

The burnout time of a rocket lifting a payload into LEO is about 600 s. Putting the relevant numbers into Equation 7.7.15 yields the result

$$\frac{m_F}{m_R + m_p} = e^{(2.67+2.00)} - 1 \approx 105$$

In other words, it takes about 105 kg of fuel to place 1 kg of *stuff* into orbit! This ratio is larger than that which is typically required. For example, the liftoff weight of the *Saturn V* was about 3.2 million kg and it could put 100,000 kg into orbit. This is a ratio of about 32 kg of fuel for every kilogram of orbital stuff. Why is our result a factor of 3 larger?

Saturn V used a more efficient, two-stage rocket to launch a payload into LEO. The tanks that hold the fuel for the first stage are jettisoned after the first stage burn is completed; thus, this now useless mass is not boosted into orbit, which greatly reduces the overall fuel requirement. Let's take a look at Equation 7.7.14 to see how this works. We denote the mass ratio by the symbol, μ

$$\left[\frac{m_R + m_p + m_F}{m_R + m_p}\right] = \mu \tag{7.7.16}$$

We assume that the mass ratio of the first stage μ_1 is equal to that of the second μ_2 and that the burnout times τ_{B1} and τ_{B2} for each stage are identical. We can then calculate the final velocities achieved by each stage from Equation 7.7.13b

$$\frac{v_{f1}}{V} = \ln\mu - \frac{\tau_B}{\tau_s} \tag{7.7.17}$$

and

$$\frac{v_{f2} - v_{f1}}{V} = \ln\mu - \frac{\tau_B}{\tau_s} \tag{7.7.18}$$

Solving for v_{f2} gives

$$\frac{v_{f2}}{V} = 2\ln\mu - 2\frac{\tau_B}{\tau_s} \tag{7.7.19}$$

Solving for the fuel to rocket and payload mass ratio as before gives

$$\frac{m_F}{m_R + m_p} = e^{\left[\frac{v_{f2}}{2V} + \frac{\tau_B}{\tau_s}\right]} - 1 \tag{7.7.20}$$

Putting in the numbers, we get

$$\frac{m_F}{m_R + m_p} \approx 27 \tag{7.7.21}$$

Thus, it takes only about 27 kg of fuel to put 1 kg of stuff into orbit using a two-stage rocket. Clearly, there is an enormous advantage to staging as was demonstrated in Saturn V.

The Ion Rocket

Chemical rockets use the thermal energy released in the explosive oxidation of the fuel in the rocket motor chamber to eject the reactant products out the rear end of the rocket to propel it forward. In an ion rocket, such as NASA's *Deep Space 1*,[9] atoms of xenon gas are stripped of one of their electrons, and the resulting positive Xe^+ ions are accelerated by an electric field in the rocket motor. These ejected ions impart a forward momentum to the rocket exactly in the same way as described by Equation 7.7.7. There are two essential differences between ion and chemical rockets:

- The exhaust velocity of an ion rocket is about 10 times larger than that of a chemical rocket, which gives a larger specific impulse (see Equation 7.7.12)
- The mass ejected per unit time, $\dot{m}$, is much smaller in ion rockets, which gives a much smaller thrust (the term $V\dot{m}$ in Equation 7.7.7).

These differences crop up because, even though the electrostatic acceleration of ions is more efficient than thermal acceleration by chemical explosions, the density of ejected ions is much less than the density of the ejected gasses. The upshot is that an ion rocket is more efficient than a chemical rocket in the sense that it takes much less fuel mass to propel the rocket to some desired speed, but the acceleration of the rocket is quite gentle, so that it takes more time to attain that speed. This makes ion rockets more suitable for deep space missions to, say, comets and asteroids and, perhaps, ultimately, to nearby star systems! Indeed, one of the purposes of NASA's Deep Space 1 mission is to test this hypothesis. Here we discuss its propulsion system to see what has been achieved so far with this new technology.

The electrostatic potential Φ_e through which the Xe^+ ions were accelerated, was 1280 volts. The ions were ejected from the 0.3-meter thruster through a pair of focusing molybdenum grids. We can estimate the maximum possible escape velocity of these ions by noting that charged particles in an electrostatic potential Φ_e accelerate and gain kinetic energy by losing electrostatic potential energy $e\Phi_e$ where e is their electric charge. The electrostatic potential energy of a charged particle in an electric field is analogous to the gravitational potential energy $m\Phi$ (Equation 6.7.6) of a particle in a gravitational field. Thus, we have

$$\tfrac{1}{2}mV^2 = e\Phi_e \tag{7.7.22}$$

[9] See the website http://nmp.jpl.nasa.gov/ds1/tech/ionpropfaq.html for a discussion of NASA's New Millennium Project, Deep Space 1 mission.

where m is the mass of a Xe^+ ion. Solving for the escape velocity, we get

$$V = \sqrt{\frac{2e\Phi_e}{m}} \tag{7.7.23}$$

Putting in numbers:[10]

$$\begin{aligned} m &\approx 131 \text{ AMU} = 131 \times 1.66 \times 10^{-27} \text{ kg} = 2.17 \times 10^{-25} \text{ kg} \\ e &= 1.6 \times 10^{-19} \text{ C} \\ V &= 4.3 \times 10^{4} \text{ m/s} \end{aligned} \tag{7.7.24}$$

Thus, the maximum possible specific impulse of the ion rocket is

$$\tau = \frac{V}{g} = \frac{4.3 \times 10^4 \text{ m/s}}{9.8 \text{ m/s}^2} = 4.4 \times 10^3 \text{ s} \tag{7.7.25}$$

In fact, the specific impulse of Deep Space I ranges between 1900 s and 3200 s depending upon throttle power. The maximum calculated here assumes that all the available power accelerates the ions with 100% efficiency and ejects them exactly in the backward direction out the rear end of the rocket, which is virtually impossible to do. The specific impulse of Deep Space I is about 10 times greater than that of Saturn V.

We now calculate the thrust of Deep Space I, again assuming that all available power is converted into the ejected ion beam with 100% efficiency. The maximum available power on Deep Space I is P = 2.5 kW. Thus, the rate, $\dot{N}$, at which Xe^+ ions are ejected can be calculated from the expression

$$P = \dot{E} = \dot{N}e\Phi_e \tag{7.7.26}$$

Because $e\Phi_e$ is the potential energy lost in accelerating a single ion, the power consumed is equal to the potential energy lost per unit time to all the accelerated ions. The rate at which mass is ejected, $\dot{m}$, is equal to the mass of each ion times $\dot{N}$. Thus,

$$\dot{m} = m\dot{N} = \frac{mP}{e\Phi_e} = \frac{(2.17 \times 10^{-25} \text{ kg})(2500 \text{ J/s})}{(1.6 \times 10^{-19} \text{ C})(1280 \text{ V})} = 2.6 \times 10^{-6} \text{ kg/s} \tag{7.7.27}$$

where we have used the fact that 1 C × 1 V = 1 J. The maximum thrust of the ion rocket is thus,

$$\text{Thrust} = V\dot{m} = (4.3 \times 10^4 \text{ m/s})(2.6 \times 10^{-6} \text{ kg/s}) = 0.114 \text{ N}$$

In fact, the maximum thrust achieved by Deep Space I is 0.092 N. We can compare this with the thrust developed by Saturn V. Saturn V ejected about 11,700 kg/s. Thus,

$$\frac{\text{Thrust(Saturn V)}}{\text{Thrust(Deep Space I)}} = \frac{V\dot{m}\text{(Saturn V)}}{V\dot{m}\text{(Deep Space I)}} = \frac{(3000 \text{ m/s})(11{,}700 \text{ kg/s})}{0.092 \text{ N}} = 3.8 \times 10^8$$

[10] An AMU is an atomic mass unit. It is equal to 1.66×10^{-27} kg. The unit of electric charge is the Coulomb (C). The charge of the electron is -1.6×10^{-19} C; thus, the charge of a singly charged positive ion is $+1.6 \times 10^{-19}$ C.

We conclude that ion rockets are not useful for launching payloads from Earth but are suitable for deep space missions starting from Earth orbit in which efficient but gentle propulsion systems can be used.

Problems

7.1 A system consists of three particles, each of unit mass, with positions and velocities as follows:

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{i}+\mathbf{j} & \mathbf{v}_1 &= 2\mathbf{i} \\ \mathbf{r}_2 &= \mathbf{j}+\mathbf{k} & \mathbf{v}_2 &= \mathbf{j} \\ \mathbf{r}_3 &= \mathbf{k} & \mathbf{v}_3 &= \mathbf{i}+\mathbf{j}+\mathbf{k} \end{aligned}$$

Find the position and velocity of the center of mass. Find also the linear momentum of the system.

7.2 (a) Find the kinetic energy of the system in Problem 7.1.
(b) Find the value of $mv_{cm}^2/2$.
(c) Find the angular momentum about the origin.

7.3 A bullet of mass m is fired from a gun of mass M. If the gun can recoil freely and the muzzle velocity of the bullet (velocity relative to the gun as it leaves the barrel) is v_0, show that the actual velocity of the bullet relative to the ground is $v_0/(1+\gamma)$ and the recoil velocity for the gun is $-\gamma v_0/(1+\gamma)$, where $\gamma = m/M$.

7.4 A block of wood rests on a smooth horizontal table. A gun is fired horizontally at the block and the bullet passes through the block, emerging with half its initial speed just before it entered the block. Show that the fraction of the initial kinetic energy of the bullet that is lost as frictional heat is $\frac{3}{4}-\frac{1}{4}\gamma$, where γ is the ratio of the mass of the bullet to the mass of the block ($\gamma < 1$).

7.5 An artillery shell is fired at an angle of elevation of 60° with initial speed v_0. At the uppermost part of its trajectory, the shell bursts into two equal fragments, one of which moves directly upward, relative to the ground, with initial speed $v_0/2$. What is the direction and speed of the other fragment immediately after the burst?

7.6 A ball is dropped from a height h onto a horizontal pavement. If the coefficient of restitution is ϵ, show that the total vertical distance the ball goes before the rebounds cease is $h(1+\epsilon^2)/(1-\epsilon^2)$. Find also the total length of time that the ball bounces.

7.7 A small car of a mass m and initial speed v_0 collides head-on on an icy road with a truck of mass $4m$ going toward the car with initial speed $\frac{1}{2}v_0$. If the coefficient of restitution in the collision is $\frac{1}{4}$, find the speed and direction of each vehicle just after colliding.

7.8 Show that the kinetic energy of a two-particle system is $\frac{1}{2}mv_{cm}^2 + \frac{1}{2}\mu v^2$, where $m = m_1 + m_2$, v is the relative speed, and μ is the reduced mass.

7.9 If two bodies undergo a direct collision, show that the loss in kinetic energy is equal to

$$\tfrac{1}{2}\mu v^2(1-\epsilon^2)$$

where μ is the reduced mass, v is the relative speed before impact, and ϵ is the coefficient of restitution.

7.10 A moving particle of mass m_1 collides elastically with a target particle of mass m_2, which is initially at rest. If the collision is head-on, show that the incident particle loses a fraction $4\mu/m$ of its original kinetic energy, where μ is the reduced mass and $m = m_1 + m_2$.