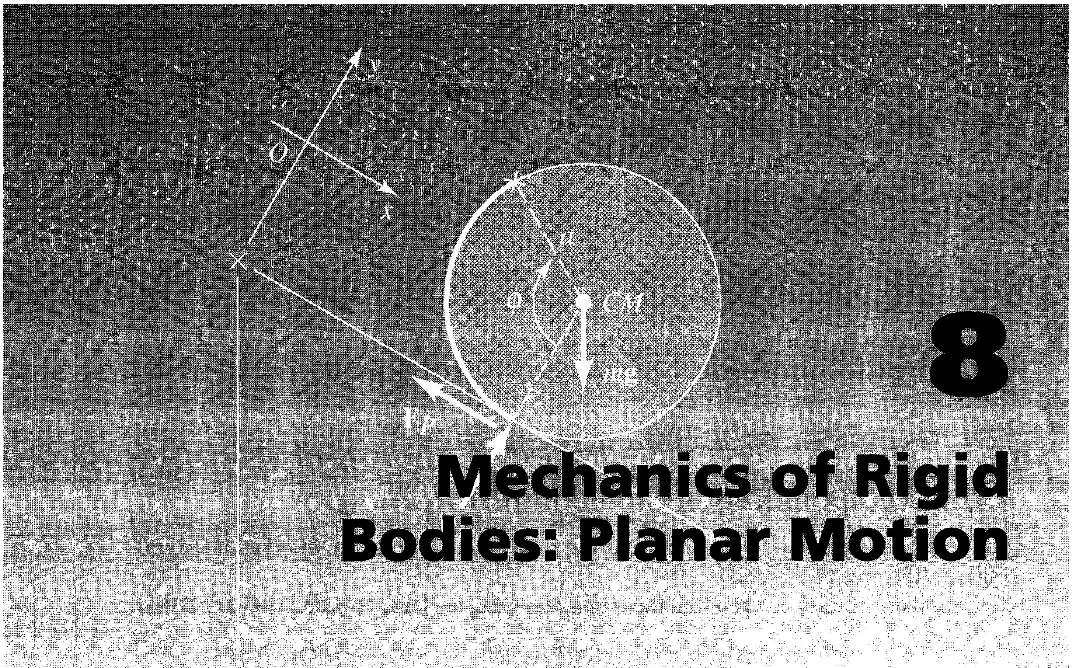


8 Mechanics of Rigid Bodies: Planar Motion

". . . centre of gravity implies the more restricted concept of a solid that is only heavy, while the centre of inertia is defined by means of the inertia alone, the forces to which the solid is subject being neglected. . . . Euler also defines the moments of inertia—a concept which Huygens lacked and which considerably simplifies the language—and calculates these moments for Homogeneous bodies."

—Rene Dugas, *A History of Mechanics,* Editions du Griffon, Neuchatel, Switzerland, 1955; synopsis of Leonhard Euler's comments in Theoria motus corporum solidorum seu rigidorum, 1760

A rigid body may be regarded as a system of particles whose *relative* positions are fixed, or, in other words, the distance between any two particles is constant. This definition of a rigid body is idealized. In the first place, as pointed out in the definition of a particle, there are no true particles in nature. Second, real extended bodies are not strictly rigid; they become more or less deformed (stretched, compressed, or bent) when external forces are applied. For the present, we shall ignore such deformations. In this chapter we take up the study of rigid-body motion for the case in which the direction of the axis of rotation does not change. The general case, which involves more extensive calculation, is treated in the next chapter.

8.1 Center of Mass of a Rigid Body

We have already defined the center of mass (Section 7.1) of a system of particles as the point (x_{cm}, y_{cm}, z_{cm}) where

$$x_{cm} = \frac{\sum_i x_i m_i}{\sum_i m_i} \qquad y_{cm} = \frac{\sum_i y_i m_i}{\sum_i m_i} \qquad z_{cm} = \frac{\sum_i z_i m_i}{\sum_i m_i} \tag{8.1.1}$$

For a rigid extended body, we can replace the summation by an integration over the volume of the body, namely,

$$x_{cm} = \frac{\int_v \rho x \, dv}{\int_v \rho \, dv} \qquad y_{cm} = \frac{\int_v \rho y \, dv}{\int_v \rho \, dv} \qquad z_{cm} = \frac{\int_v \rho z \, dv}{\int_v \rho \, dv} \tag{8.1.2}$$

where ρ is the density and dv is the element of volume.

If a rigid body is in the form of a thin shell, the equations for the center of mass become

$$x_{cm} = \frac{\int_s \rho x \, ds}{\int_s \rho \, ds} \qquad y_{cm} = \frac{\int_s \rho y \, ds}{\int_s \rho \, ds} \qquad z_{cm} = \frac{\int_s \rho z \, ds}{\int_s \rho \, ds} \tag{8.1.3}$$

where ds is the element of area and ρ is the mass per unit area, the integration extending over the area of the body.

Similarly, if the body is in the form of a thin wire, we have

$$x_{cm} = \frac{\int_l \rho x \, dl}{\int_l \rho \, dl} \qquad y_{cm} = \frac{\int_l \rho y \, dl}{\int_l \rho \, dl} \qquad z_{cm} = \frac{\int_l \rho z \, dl}{\int_l \rho \, dl} \tag{8.1.4}$$

In this case, ρ is the mass per unit length and dl is the element of length.

For uniform homogeneous bodies, the density factors ρ are constant in each case and, therefore, may be canceled out in each of the preceding equations.

If a body is composite, that is, if it consists of two or more parts whose centers of mass are known, then it is clear, from the definition of the center of mass, that we can write

$$x_{cm} = \frac{x_1 m_1 + x_2 m_2 + \cdots}{m_1 + m_2 + \cdots} \tag{8.1.5}$$

with similar equations for y_{cm} and z_{cm}. Here (x_1, y_1, z_1) is the center of mass of the part m_1, and so on.

Symmetry Considerations

If a body possesses symmetry, it is possible to take advantage of that symmetry in locating the center of mass. Thus, if the body has a plane of symmetry, that is, if each particle m_i has a mirror image of itself m_i' relative to some plane, then the center of mass lies in that plane. To prove this, let us suppose that the xy plane is a plane of symmetry. We have then

$$z_{cm} = \frac{\sum_i (z_i m_i + z_i' m_i')}{\sum_i (m_i + m_i')} \tag{8.1.6}$$

But $m_i = m_i'$ and $z_i = -z_i'$. Hence, the terms in the numerator cancel in pairs, and so $z_{cm} = 0$; that is, the center of mass lies in the xy plane.

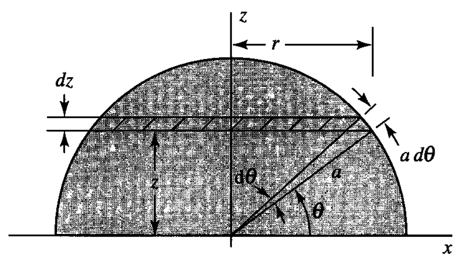


Figure 8.1.1 Coordinates for calculating the center of mass of a hemisphere.

Similarly, if the body has a line of symmetry, it is easy to show that the center of mass lies on that line. The proof is left as an exercise.

Solid Hemisphere

To find the center of mass of a solid homogeneous hemisphere of radius a, we know from symmetry that the center of mass lies on the radius that is normal to the plane face. Choosing coordinate axes as shown in Figure 8.1.1, we see that the center of mass lies on the z-axis. To calculate z_{cm} we use a circular element of volume of thickness dz and radius $= (a^2 - z^2)^{1/2}$, as shown. Thus,

$$dv = \pi(a^2 - z^2)dz \tag{8.1.7}$$

Therefore,

$$z_{cm} = \frac{\int_0^a \rho\pi z(a^2 - z^2)dz}{\int_0^a \rho\pi(a^2 - z^2)dz} = \tfrac{3}{8}a \tag{8.1.8}$$

Hemispherical Shell

For a hemispherical shell of radius a, we use the same axes as in Figure 8.1.1. Again, from symmetry, the center of mass is located on the z-axis. For our element of surface ds, we choose a circular strip of width $dl = a d\theta$. Hence,

$$ds = 2\pi r\, dl = 2\pi(a^2 - z^2)^{1/2} a\, d\theta$$

$$\theta = \sin^{-1}\left(\frac{z}{a}\right) \qquad d\theta = (a^2 - z^2)^{-1/2} dz \tag{8.1.9}$$

$$\therefore ds = 2\pi a\, dz$$

The location of the center of mass is accordingly given by

$$z_{cm} = \frac{\int_0^a \rho 2\pi a z\, dz}{\int_0^a \rho 2\pi a\, dz} = \tfrac{1}{2}a \tag{8.1.10}$$

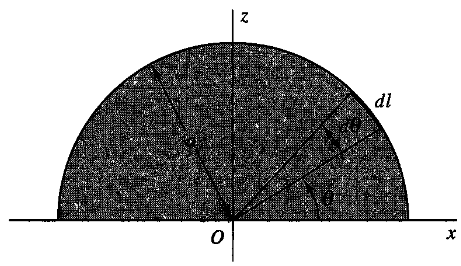


Figure 8.1.2 Coordinates for calculating the center of mass of a wire bent into the form of a semicircle.

Semicircle

To find the center of mass of a thin wire bent into the form of a semicircle of radius a, we use axes as shown in Figure 8.1.2. We have

$$dl = a\,d\theta \tag{8.1.11}$$

and

$$z = a \sin\theta \tag{8.1.12}$$

Hence,

$$z_{cm} = \frac{\int_0^{\pi} \rho(a \sin\theta) a\,d\theta}{\int_0^{\pi} \rho a\,d\theta} = \frac{2a}{\pi} \tag{8.1.13}$$

Semicircular Lamina

In the case of a uniform semicircular lamina, the center of mass is on the z-axis (Figure 8.1.2). As an exercise, the student should verify that

$$z_{cm} = \frac{4a}{3\pi} \tag{8.1.14}$$

Solid Cone of Variable Density: Numerical Integration

Sometimes we are confronted with the unfortunate prospect of having to find the center of mass of a body whose density is not uniform. In such cases, we must resort to numerical integration. Here we present a moderately complex case that we will solve numerically even though it can be solved analytically. We do this to illustrate how such a calculation can be easily carried out using the tools available in *Mathematica*.

Consider a solid, "unit" cone bounded by the conical surface $z^2 = x^2 + y^2$ and the plane $z = 1$ as shown in Figure 8.1.3, whose mass density function is given by

$$\rho(x, y, z) = \sqrt{x^2 + y^2} \tag{8.1.15}$$

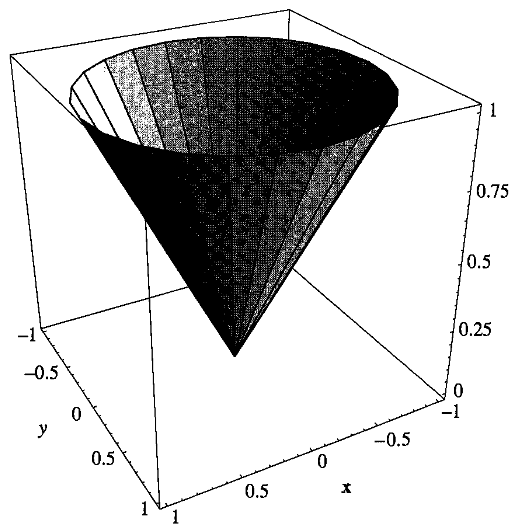


Figure 8.1.3 Solid cone whose surface is given by the curve $z^2 = x^2 + y^2$ and $z = 1$.

The center of mass of this cone can be calculated by solving the integrals given in Equation 8.1.2. The mass of the cone is given by

$$M = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} \sqrt{x^2+y^2}\; dz\,dy\,dx \tag{8.1.16}$$

Notice that the limits of integration over the variable y depend on x and that the limits of integration over z depend on both x and y. Because this integral is symmetric about both the x and y axes, it simplifies to

$$M = 4\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} \sqrt{x^2+y^2}\; dz\,dy\,dx \tag{8.1.17}$$

The first moments of the mass are given by the integrals

$$M_{yz} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} x\sqrt{x^2+y^2}\; dz\,dy\,dx \tag{8.1.18a}$$

$$M_{xz} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} y\sqrt{x^2+y^2}\; dz\,dy\,dx \tag{8.1.18b}$$

$$M_{xy} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} z\sqrt{x^2+y^2}\; dz\,dy\,dx \tag{8.1.18c}$$

The location of the center of mass is then

$$(x_{cm}, y_{cm}, z_{cm}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right) \tag{8.1.19}$$

where ϕ_i is defined as shown in Figure 8.2.1. Equations 8.2.2 can also be obtained by extracting the components of the vector equation

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i \tag{8.2.3}$$

where $\boldsymbol{\omega} = \mathbf{k}\omega$.

Let us calculate the kinetic energy of rotation of the body. We have

$$T_{rot} = \sum_i \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \tfrac{1}{2} I_z \omega^2 \tag{8.2.4}$$

where

$$I_z = \sum_i m_i r_i^2 = \sum_i m_i\left(x_i^2 + y_i^2\right) \tag{8.2.5}$$

The quantity I_z, defined by Equation 8.2.5, is called the *moment of inertia* about the z-axis.

To show how the moment of inertia further enters the picture, let us next calculate the angular momentum about the axis of rotation. Because the angular momentum of a single particle is, by definition, $\mathbf{r}_i \times m_i\mathbf{v}_i$, the z-component is

$$m_i(x_i\dot{y}_i - y_i\dot{x}_i) = m_i\left(x_i^2 + y_i^2\right)\omega = m_i r_i^2 \omega \tag{8.2.6}$$

where we have made use of Equations 8.2.2. The total z-component of the angular momentum, which we call L_z, is then given by summing over all the particles, namely,

$$L_z = \sum_i m_i r_i^2 \omega = I_z \omega \tag{8.2.7}$$

In Section 7.2 we found that the rate of change of angular momentum for any system is equal to the total moment of the external forces. For a body constrained to rotate about a fixed axis, taken here as the z-axis, then

$$N_z = \frac{dL_z}{dt} = \frac{d(I_z\omega)}{dt} \tag{8.2.8}$$

where N_z is the total moment of all the applied forces about the axis of rotation (the component of $\mathbf{N}$ along the z-axis). If the body is rigid, then I_z is constant, and we can write

$$N_z = I_z \frac{d\omega}{dt} \tag{8.2.9}$$

The analogy between the equations for translation and for rotation about a fixed axis is shown in the following table:

Translation along x-axis		***Rotation about z-axis***	
Linear momentum	$p_x = mv_x$	Angular momentum	$L_z = I_z\omega$
Force	$F_x = m\dot{v}_x$	Torque	$N_z = I_z\dot{\omega}$
Kinetic energy	$T = \frac{1}{2}mv^2$	Kinetic energy	$T_{rot} = \frac{1}{2}I_z\omega^2$

Thus, the moment of inertia is analogous to mass; it is a measure of the rotational inertia of a body relative to some fixed axis of rotation, just as mass is a measure of translational inertia of a body.

8.3 Calculation of the Moment of Inertia

In calculations of the moment of inertia $\Sigma\, m_i r_i^2$ for extended bodies, we can replace the summation by an integration over the body, just as we did in calculation of the center of mass. Thus, we may write for any axis

$$I = \int r^2 dm \tag{8.3.1}$$

where the element of mass dm is given by a density factor multiplied by an appropriate differential (volume, area, or length), and r is the perpendicular distance from the element of mass to the axis of rotation.[1]

In the case of a composite body, from the definition of the moment of inertia, we may write

$$I = I_1 + I_2 + \cdots \tag{8.3.2}$$

where I_1, I_2, and so on, are the moments of inertia of the various parts about the particular axis chosen.

Let us calculate the moments of inertia for some important special cases.

Thin Rod

For a thin, uniform rod of length a and mass m, we have, for an axis perpendicular to the rod at one end (Figure 8.3.1a),

$$I_z = \int_0^a x^2 \rho dx = \tfrac{1}{3}\rho a^3 = \tfrac{1}{3} ma^2 \tag{8.3.3}$$

The last step follows from the fact that $m = \rho a$.

If the axis is taken at the center of the rod (Figure 8.3.1b), we have

$$I_z = \int_{-a/2}^{a/2} x^2 \rho\, dx = \tfrac{1}{12}\rho a^3 = \tfrac{1}{12} ma^2 \tag{8.3.4}$$

Hoop or Cylindrical Shell

In the case of a thin circular hoop or cylindrical shell, for the central, or *symmetry*, axis, all particles lie at the same distance from the axis. Thus,

$$I_{axis} = ma^2 \tag{8.3.5}$$

where a is the radius and m is the mass.

[1]In Chapter 9, when we discuss the rotational motion of three-dimensional bodies, the distance between the mass element dm and the axis of rotation $r_\perp$ is designed to remind us that the relevant distance is the one perpendicular to the axis of rotation.

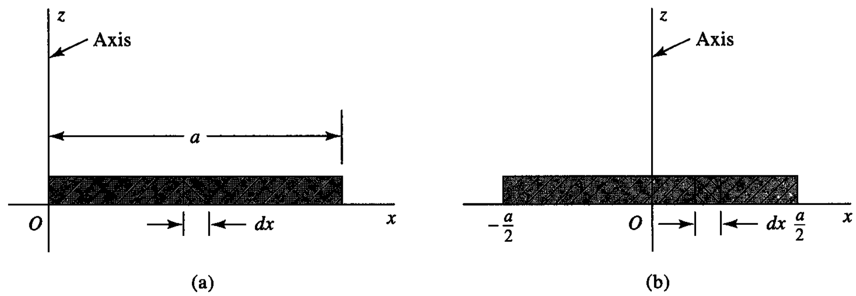


Figure 8.3.1 Coordinates for calculating the moment of inertia of a rod (a) about one end and (b) about the center of the rod.

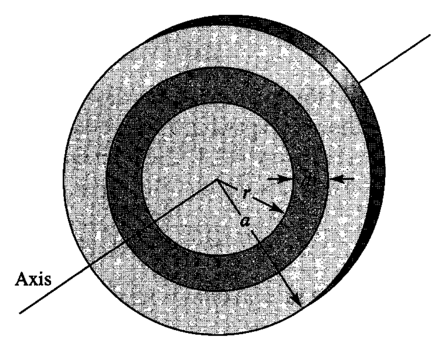


Figure 8.3.2 Coordinates for finding the moment of inertia of a disc.

Circular Disc or Cylinder

To calculate the moment of inertia of a uniform circular disc of radius a and mass m, we use polar coordinates. The element of mass, a thin ring of radius r and thickness dr, is given by

$$dm = \rho 2\pi r\, dr \qquad (8.3.6)$$

where ρ is the mass per unit area. The moment of inertia about an axis through the center of the disc normal to the plane faces (Figure 8.3.2) is obtained as follows:

$$I_{axis} = \int_0^a r^2 \rho\, 2\pi r\, dr = 2\pi\rho \frac{a^4}{4} = \tfrac{1}{2} ma^2 \qquad (8.3.7)$$

The last step results from the relation $m = \rho \pi a^2$.

Equation 8.3.7 also applies to a uniform right-circular cylinder of radius a and mass m, the axis being the central axis of the cylinder.

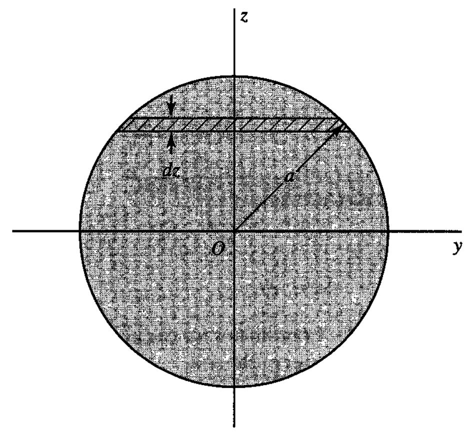


Figure 8.3.3 Coordinates for finding the moment of inertia of a sphere.

Sphere

Let us find the moment of inertia of a uniform solid sphere of radius a and mass m about an axis (the z-axis) passing through the center. We divide the sphere into thin circular discs, as shown in Figure 8.3.3. The moment of inertia of a representative disc of radius y, from Equation 8.3.7, is $\frac{1}{2}y^2 dm$. But $dm = \rho\pi y^2\, dz$; hence,

$$I_z = \int_{-a}^{a} \frac{1}{2}\pi\rho y^4\, dz = \int_{-a}^{a} \frac{1}{2}\pi\rho(a^2 - z^2)^2\, dz = \frac{8}{15}\pi\rho a^5 \tag{8.3.8}$$

The last step in Equation 8.3.8 should be filled in by the student. Because the mass m is given by

$$m = \frac{4}{3}\pi a^3 \rho \tag{8.3.9}$$

we have

$$I_z = \frac{2}{5}ma^2 \tag{8.3.10}$$

for a solid uniform sphere. Clearly also, $I_x = I_y = I_z$.

Spherical Shell

The moment of inertia of a thin, uniform, spherical shell can be found very simply by application of Equation 8.3.8. If we differentiate with respect to a, namely,

$$dI_z = \frac{8}{3}\pi\rho a^4\, da \tag{8.3.11}$$

the result is the moment of inertia of a shell of thickness da and radius a. The mass of the shell is $4\pi a^2 \rho\, da$. Hence, we can write

$$I_z = \frac{2}{3}ma^2 \tag{8.3.12}$$

for the moment of inertia of a thin shell of radius a and mass m. The student should verify this result by direct integration.

EXAMPLE 8.3.1

Shown in Figure 8.3.4 is a uniform chain of length $l = 2\pi R$ and mass $m = M/2$ that is initially wrapped around a uniform, thin disc of radius R and mass M. One tiny piece of chain initially hangs free, perpendicular to the horizontal axis. When the disc is released, the chain falls and unwraps. The disc begins to rotate faster and faster about its fixed z-axis, without friction. (a) Find the angular speed of the disc at the moment the chain completely unwraps. (b) Solve for the case of a chain wrapped around a wheel whose mass is the same as that of the disc but concentrated in a thin rim.

Solution:

(a) Figure 8.3.4 shows the disc and chain at the moment the chain unwrapped. The final angular speed of the disc is ω. Energy was conserved as the chain unwrapped. Because the center of mass of the chain originally coincided with that of the disc, it fell a distance $l/2 = \pi R$, and we have

$$mg\frac{l}{2} = \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}mv^2$$

$$\frac{l}{2} = \pi R \qquad v = \omega R \qquad I = \tfrac{1}{2}MR^2$$

Solving for ω^2 gives

$$\omega^2 = \frac{mg(l/2)}{\left[\left(\frac{1}{2}\right)(M/2) + \left(\frac{1}{2}\right)(m)\right]R^2} = \frac{mg\pi R}{\left[\left(\frac{1}{2}\right)m + \left(\frac{1}{2}\right)m\right]R^2}$$

$$= \pi\frac{g}{R}$$

(b) The moment of inertia of a wheel is $I = MR^2$. Substituting this into the preceding equation yields

$$\omega^2 = \pi\frac{2g}{3R}$$

Even though the mass of the wheel is the same as that of the disc, its moment of inertia is larger, because all its mass is concentrated along the rim. Thus, its angular acceleration and final angular velocity are less than that of the disc.

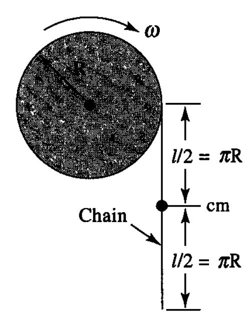


Figure 8.3.4 Falling chain attached to disc, free to rotate about a fixed z-axis.

Perpendicular-Axis Theorem for a Plane Lamina

Consider a rigid body that is in the form of a plane lamina of any shape. Let us place the lamina in the xy plane (Figure 8.3.5). The moment of inertia about the z-axis is given by

$$I_z = \sum_i m_i\left(x_i^2 + y_i^2\right) = \sum_i m_i x_i^2 + \sum_i m_i y_i^2 \tag{8.3.13}$$

The sum $\Sigma\, m_i x_i^2$ is just the moment of inertia I_y about the y-axis, because z_i is zero for all particles. Similarly, $\Sigma_i\, m_i y_i^2$ is the moment of inertia I_x about the x-axis. Equation 8.3.13 can, therefore, be written

$$I_z = I_x + I_y \tag{8.3.14}$$

This is the perpendicular-axis theorem. In words:

> The moment of inertia of any plane lamina about an axis normal to the plane of the lamina is equal to the sum of the moments of inertia about any two mutually perpendicular axes passing through the given axis and lying in the plane of the lamina.

As an example of the use of this theorem, let us consider a thin circular disc in the xy plane (Figure 8.3.6). From Equation 8.3.7 we have

$$I_z = \tfrac{1}{2}ma^2 = I_x + I_y \tag{8.3.15}$$

In this case, however, we know from symmetry that $I_x = I_y$. We must, therefore, have

$$I_x = I_y = \tfrac{1}{4}ma^2 \tag{8.3.16}$$

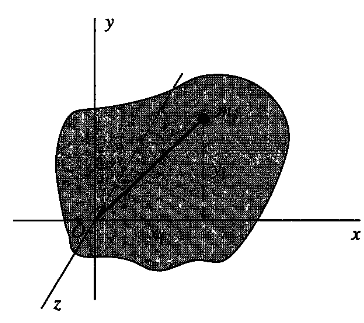


Figure 8.3.5 The perpendicular-axis theorem for a lamina.

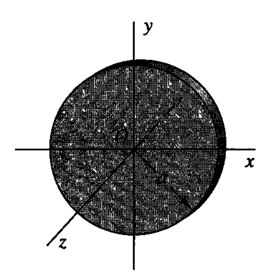


Figure 8.3.6 Circular disc.

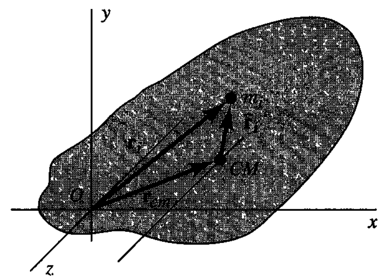


Figure 8.3.7 The parallel-axis theorem for any rigid body.

for the moment of inertia about any axis in the plane of the disc passing through the center. This result can also be obtained by direct integration.

Parallel-Axis Theorem for Any Rigid Body

Consider the equation for the moment of inertia about some axis, say the z-axis,

$$I_z = \sum_i m_i\left(x_i^2 + y_i^2\right) \tag{8.3.17}$$

Now we can express x_i and y_i in terms of the coordinates of the center of mass (x_{cm}, y_{cm}, z_{cm}) and the coordinates *relative* to the center of mass $(\bar{x}_i, \bar{y}_i, \bar{z}_i)$ (Figure 8.3.7) as follows:

$$x_i = x_{cm} + \bar{x}_i \qquad y_i = y_{cm} + \bar{y}_i \tag{8.3.18}$$

We have, therefore, after substituting and collecting terms,

$$I_z = \sum_i m_i\left(\bar{x}_i^2 + \bar{y}_i^2\right) + \sum_i m_i\left(x_{cm}^2 + y_{cm}^2\right) + 2x_{cm}\sum_i m_i\bar{x}_i + 2y_{cm}\sum_i m_i\bar{y}_i \tag{8.3.19}$$

The first sum on the right is just the moment of inertia about an axis parallel to the z-axis and passing through the center of mass. We call it I_{cm}. The second sum is equal to the mass of the body multiplied by the square of the distance between the center of mass and the z-axis. Let us call this distance l. That is, $l^2 = x_{cm}^2 + y_{cm}^2$.

Now, from the definition of the center of mass,

$$\sum_i m_i\bar{x}_i = \sum_i m_i\bar{y}_i = 0 \tag{8.3.20}$$

Hence, the last two sums on the right of Equation 8.3.19 vanish. The final result may be written in the general form for any axis

$$I = I_{cm} + ml^2 \tag{8.3.21}$$

This is the *parallel-axis* theorem. It is applicable to any rigid body, solid as well as laminar.

The theorem states, in effect, that:

> The moment of inertia of a rigid body about any axis is equal to the moment of inertia about a parallel axis passing through the center of mass plus the product of the mass of the body and the square of the distance between the two axes.

We can use the parallel-axis theorem to calculate the moment of inertia of a uniform circular disc about an axis *perpendicular* to the plane of the disc and passing through an edge (see Figure 8.3.8a). Using Equations 8.3.7 and 8.3.21, we get

$$I = \tfrac{1}{2}ma^2 + ma^2 = \tfrac{3}{2}ma^2 \tag{8.3.22}$$

We can also use the parallel-axis theorem to calculate the moment of inertia of the disc about an axis *in the plane* of the disc and *tangent* to an edge (see Figure 8.3.8b). Using Equations 8.3.16 and 8.3.21, we get

$$I = \tfrac{1}{4}ma^2 + ma^2 = \tfrac{5}{4}ma^2 \tag{8.3.23}$$

As a second example, let us find the moment of inertia of a uniform circular cylinder of length b and radius a about an axis through the center and *perpendicular* to the central axis, namely I_x or I_y in Figure 8.3.9. For our element of integration, we choose a disc of thickness dz located a distance z from the xy plane. Then, from the previous result for a thin disc (Equation 8.3.16), together with the parallel-axis theorem, we have

$$dI_x = \tfrac{1}{4}a^2 dm + z^2 dm \tag{8.3.24}$$

in which $dm = \rho\pi a^2\, dz$. Thus,

$$I_x = \rho\pi a^2 \int_{-b/2}^{b/2} \left(\tfrac{1}{4}a^2 + z^2\right)dz = \rho\pi a^2\left(\tfrac{1}{4}a^2 b + \tfrac{1}{12}b^3\right) \tag{8.3.25}$$

But the mass of the cylinder is $m = \rho\pi a^2 b$, therefore,

$$I_x = I_y = m\left(\tfrac{1}{4}a^2 + \tfrac{1}{12}b^2\right) \tag{8.3.26}$$

Radius of Gyration

Note the similarity of Equation 8.2.5, the expression for the moment of inertia I_z of a rigid body about the z-axis, to the expressions for center of mass developed in Section 8.1. If we were to divide Equation 8.2.5 by the total mass of the rigid body, we would obtain the mass-weighted average of the square of the positions of all the mass elements away from the z-axis.

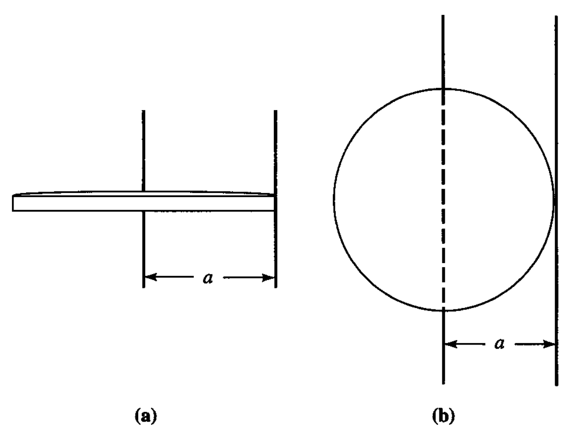


Figure 8.3.8 Moment of inertia of a uniform, thin disc about axes (a) perpendicular to the plane of the disc and through an edge and (b) in the plane of the disc and tangent to an edge.

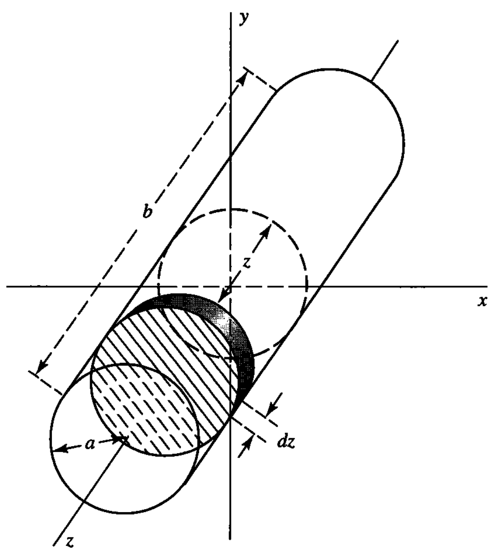


Figure 8.3.9 Coordinates for finding the moment of inertia of a circular cylinder.

Thus, moment of inertia is, in essence, the average of the squares of the radial distances away from the z-axis of all the mass elements making up the rigid body. You can understand physically why the moment of inertia must depend on the square (or, at least, some even power) of the distances away from the rotational axis; it could not be represented by a linear average over all the mass elements (or any average of the odd power of distance). If such were the case, then a body whose mass was symmetrically distributed about its rotational axis, such as a bicycle wheel, would have zero moment of inertia because of a term-by-term cancellation of the positive and negative weighted mass elements in the symmetrical distribution. An application of the slightest torque would spin up a bicycle wheel into an instantaneous frenzy, a condition that any bike racer knows is impossible.

We can formalize this discussion by defining a distance k, called the *radius of gyration*, to be this average, that is,

$$k^2 = \frac{I}{m} \qquad k = \sqrt{\frac{I}{m}} \tag{8.3.27}$$

Knowing the radius of gyration of any rigid body is equivalent to knowing its moment of inertia, but it better characterizes the nature of the averaging process on which the concept of moment of inertia is based.

For example, we find for the radius of gyration of a thin rod about an axis passing through one end (see Equation 8.3.3)

$$k = \sqrt{\frac{\left(\frac{1}{3}\right)ma^2}{m}} = \frac{a}{\sqrt{3}} \tag{8.3.28}$$

Moments of inertia for various objects can be tabulated simply by listing the squares of their radii of gyration (Table 8.3.1).

TABLE 8.3.1 Values of k^2 of Various Bodies (Moment of Inertia = Mass × k^2)

Body	Axis	k^2
Thin rod, lenght a	Normal to rod at its center	$\frac{a^2}{12}$
	Normal to rod at one end	$\frac{a^2}{3}$
Thin rectangular lamina, sides a and b	Through the center, parallel to side b	$\frac{a^2}{12}$
	Through the center, normal to the lamina	$\frac{a^2+b^2}{12}$
Thin circular disc, radius a	Through the center, in the plane of the disc	$\frac{a^2}{4}$
	Through the center, normal to the disc	$\frac{a^2}{2}$
Thin hoop (or ring), radius a	Through the center, in the plane of the hoop	$\frac{a^2}{2}$
	Through the center, normal to the plane of the hoop	a^2
Thin cylindrical shell, radius a, length b	Central longitudinal axis	a^2
Uniform solid right circular cylinder, radius a, length b	Central longitudinal axis	$\frac{a^2}{2}$
	Through the center, perpendicular to longitudinal axis	$\frac{a^2}{4}+\frac{b^2}{12}$
Thin spherical shell, radius a	Any diameter	$\frac{2}{3}a^2$
Uniform solid sphere, radius a	Any diameter	$\frac{2}{5}a^2$
Uniform solid rectangular parallelepiped, sides a, b, and c	Through the center, normal to face ab, parallel to edge c	$\frac{a^2+b^2}{12}$

8.4 The Physical Pendulum

A rigid body that is free to swing under its own weight about a fixed horizontal axis of rotation is known as a *physical pendulum*, or *compound pendulum*. A physical pendulum is shown in Figure 8.4.1. Here CM is the center of mass, and O is the point on the axis of rotation that is in the vertical plane of the circular path of the center of mass.

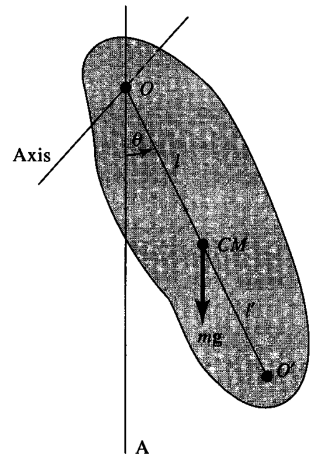


Figure 8.4.1 The physical pendulum.

Denoting the angle between the line OCM and the vertical line OA by θ, the moment of the gravitational force (acting at CM) about the axis of rotation is of magnitude

$$mgl \sin \theta$$

The fundamental equation of motion $N = I\dot{\omega}$ then takes the form $-mgl \sin \theta = I\ddot{\theta}$

$$\ddot{\theta} + \frac{mgl}{I} \sin\theta = 0 \tag{8.4.1}$$

Equation 8.4.1 is identical in form to the equation of motion of a simple pendulum. For small oscillations, as in the case of the simple pendulum, we can replace $\sin\theta$ by θ:

$$\ddot{\theta} + \frac{mgl}{I}\theta = 0 \tag{8.4.2}$$

The solution, as we know from Chapter 3, can be written

$$\theta = \theta_0 \cos(2\pi f_0 t - \delta) \tag{8.4.3}$$

where θ_0 is the amplitude and δ is a phase angle. The frequency of oscillation is given by

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{mgl}{I}} \tag{8.4.4}$$

The period is, therefore, given by

$$T_0 = \frac{1}{f_0} = 2\pi\sqrt{\frac{I}{mgl}} \tag{8.4.5}$$

(To avoid confusion, we have used the frequency f_0 instead of the angular frequency ω_0 to characterize the oscillation of the pendulum.) We can also express the period in terms of the radius of gyration k, namely,

$$T_0 = 2\pi\sqrt{\frac{k^2}{gl}} \tag{8.4.6}$$

Thus, the period is the same as that of a simple pendulum of length k^2/l.

Consider as an example a thin uniform rod of length a swinging as a physical pendulum about one end: $k^2 = a^2/3$, $l = a/2$. The period is then

$$T_0 = 2\pi\sqrt{\frac{a^2/3}{ga/2}} = 2\pi\sqrt{\frac{2a}{3g}} \tag{8.4.7}$$

which is the same as that of a simple pendulum of length $\frac{2}{3}a$.

Center of Oscillation

By use of the parallel-axis theorem, we can express the radius of gyration k in terms of the radius of gyration about the center of mass k_{cm}, as follows:

$$I = I_{cm} + ml^2 \tag{8.4.8}$$

or

$$mk^2 = mk_{cm}^2 + ml^2 \tag{8.4.9a}$$

Canceling the m's, we get

$$k^2 = k_{cm}^2 + l^2 \tag{8.4.9b}$$

Equation 8.4.6 can, therefore, be written as

$$T_0 = 2\pi\sqrt{\frac{k_{cm}^2 + l^2}{gl}} \tag{8.4.10}$$

Suppose that the axis of rotation of a physical pendulum is shifted to a different position O' at a distance l' from the center of mass, as shown in Figure 8.4.1. The period of oscillation T_0' about this new axis is given by

$$T_0' = 2\pi\sqrt{\frac{k_{cm}^2 + l'^2}{gl'}} \tag{8.4.11}$$

The periods of oscillation about O and about O' are equal, provided

$$\frac{k_{cm}^2 + l^2}{l} = \frac{k_{cm}^2 + l'^2}{l'} \tag{8.4.12}$$

Equation 8.4.12 readily reduces to

$$ll' = k_{cm}^2 \tag{8.4.13}$$

The point O', related to O by Equation 8.4.13, is called the *center of oscillation* for the point O. O is also the center of oscillation for O'. Thus, for a rod of length a swinging about one end, we have $k_{cm}^2 = a^2/12$ and $l = a/2$. Hence, from Equation 8.4.13, $l' = a/6$, and so the rod has the same period when swinging about an axis located a distance $a/6$ from the center as it does for an axis passing through one end.

The "Upside-Down Pendulum": Elliptic Integrals

When the amplitude of oscillation of a pendulum is so large that the approximation $\sin\theta = \theta$ is not valid, the formula for the period (Equation 8.4.5) is not accurate. In Example 3.7.1 we obtained an improved formula for the period of a simple pendulum by using a method of successive approximations. That result also applies to the physical pendulum with l replaced by I/ml, but it is still an approximation and is completely erroneous when the amplitude approaches 180° (vertical position) (Figure 8.4.2).

To find the period for large amplitude, we start with the energy equation for the physical pendulum

$$\tfrac{1}{2}I\dot{\theta}^2 + mgh = E \tag{8.4.14}$$

where h is the vertical distance of the center of mass from the equilibrium position, that is, $h = l(1 - \cos\theta)$. Let θ_0 denote the amplitude of the pendulum's oscillation. Then $\dot{\theta} = 0$ when $\theta = \theta_0$, so that $E = mgl(1 - \cos\theta_0)$. The energy equation can then be written

$$\tfrac{1}{2}I\dot{\theta}^2 + mgl(1-\cos\theta) = mgl(1-\cos\theta_0) \tag{8.4.15}$$

Solving for $\dot{\theta}$ gives

$$\frac{d\theta}{dt} = \pm\left[\frac{2mgl}{I}(\cos\theta - \cos\theta_0)\right]^{1/2} \tag{8.4.16}$$

Thus, by taking the positive root, we can write

$$t = \sqrt{\frac{I}{2mgl}}\int_0^\theta \frac{d\theta}{(\cos\theta - \cos\theta_0)^{1/2}} \tag{8.4.17}$$

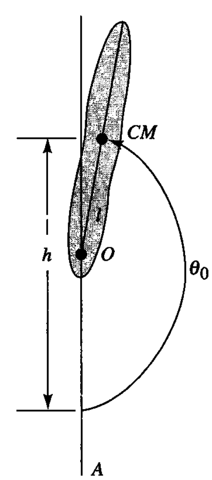


Figure 8.4.2 The upside-down pendulum.

from which we can, in principle, find t as a function of θ. Also, we note that θ increases from 0 to θ_0 in just one quarter of a complete cycle. The period T can, therefore, be expressed as

$$T = 4\sqrt{\frac{I}{2mgl}}\int_0^{\theta_0}\frac{d\theta}{(\cos\theta - \cos\theta_0)^{1/2}} \tag{8.4.18}$$

Unfortunately, the integrals in Equations 8.4.17 and 8.4.18 cannot be evaluated in terms of elementary functions. They can, however, be expressed in terms of special functions known as *elliptic integrals.* For this purpose it is convenient to introduce a new variable of integration ϕ, which is defined as follows:

$$\sin\phi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)} = \frac{1}{k}\sin\left(\frac{\theta}{2}\right) \tag{8.4.19}$$

where[2]

$$k = \sin\left(\frac{\theta_0}{2}\right)$$

Thus, when $\theta = \theta_0$, we have $\sin\phi = 1$ and so $\phi = \pi/2$. The result of making these substitutions in Equations 8.4.17 and 8.4.18 yields

$$t = \sqrt{\frac{I}{mgl}}\int_0^{\phi}\frac{d\phi}{(1 - k^2\sin^2\phi)^{1/2}} \tag{8.4.20a}$$

$$T = 4\sqrt{\frac{I}{mgl}}\int_0^{\pi/2}\frac{d\phi}{(1 - k^2\sin^2\phi)^{1/2}} \tag{8.4.20b}$$

The steps are left as an exercise and involve use of the identity $\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$.

Tabulated values of the integrals in the preceding expressions can be found in various handbooks and mathematical tables. The first integral

$$\int_0^{\phi}\frac{d\phi}{(1 - k^2\sin^2\phi)^{1/2}} = F(k,\phi) \tag{8.4.21}$$

is called the *incomplete elliptic integral of the first kind.* In our problem, given a value of the amplitude θ_0, we can find the relationship between θ and t through a series of steps involving the definitions of k and ϕ. We are more interested in finding the period of the pendulum, which involves the second integral

$$\int_0^{\pi/2}\frac{d\phi}{(1 - k^2\sin^2\phi)^{1/2}} = F\left(k,\frac{\pi}{2}\right) \tag{8.4.22}$$

[2]Note that k defined here is a parameter that characterizes elliptic integrals. It is not the k defined previously as the radius of gyration.

TABLE 8.4.1 Selected Values of the Complete Elliptic Integral and Corresponding Period of Oscillation of a Physical Pendulum[1]

Amplitude, θ_0	$k = \sin\left(\frac{\theta_0}{2}\right)$	$F\left(k, \frac{\pi}{2}\right)$	Period, T
0°	0	$1.5708 = \pi/2$	T_0
10°	0.0872	1.5738	$1.0019\ T_0$
45°	0.3827	1.6336	$1.0400\ T_0$
90°	0.7071	1.8541	$1.1804\ T_0$
135°	0.9234	2.4003	$1.5281\ T_0$
178°	0.99985	5.4349	$3.5236\ T_0$
179°	0.99996	5.2660	$4.6002\ T_0$
180°	1	∞	∞

[1] For more extensive tables and other information on elliptic integrals, consult any treatise on elliptic functions, such as (1) H. B. Dwight, *Tables of Integrals and Other Mathematical Data*, The Macmillan Co., New York, 1961; and (2) M. Abramowitz and A. Stegun, *Handbook of Mathematical Functions*, Dover Publishing, New York, 1972.

known as the *complete elliptic integral of the first kind.* (It is also variously listed as $K(k)$ or $F(k)$ in many tables.) In terms of it, the period is

$$T = 4\sqrt{\frac{I}{mgl}}\,F\left(k, \frac{\pi}{2}\right) \tag{8.4.23}$$

Table 8.4.1 lists selected values of $F(k, \pi/2)$. Also listed is the period T as a factor multiplied by the period for zero amplitude: $T_0 = 2\pi(I/mgl)^{1/2}$.

Table 8.4.1 shows the trend as the amplitude approaches 180° at which value the elliptic integral diverges and the period becomes infinitely large. This means that, *theoretically*, a physical pendulum, such as a rigid rod, if placed exactly in the vertical position with absolutely zero initial angular velocity, would remain in that same unstable position indefinitely.

EXAMPLE 8.4.1

A physical pendulum, as shown in Figure 8.4.1, is hanging vertically at rest. It is struck a sudden blow such that its total energy after the blow is $E = 2mgl$, where m is the mass of the pendulum and l is the distance of its center of mass to the pivot point. (a) Calculate the angle of displacement θ away from the vertical as a function of time. (b) Does the pendulum reach the "upside-down" configuration, $\theta = \pi$? If so, use your result from part (a) to calculate how long it takes.

Solution:

(a) We begin by writing down the total energy of the pendulum as in Equation 8.4.15

$$\tfrac{1}{2}I\dot{\theta}^2 + mgl(1 - \cos\theta) = 2mgl$$

Solving for $\dot{\theta}^2$

$$\dot{\theta}^2 = \frac{2mgl}{I}(1+\cos\theta) = \frac{4mgl}{I}\cos^2\frac{\theta}{2}$$

We introduce the following substitution, $y = \sin\theta/2$, to eliminate integrals involving trigonometric functions and obtain a moderately simple analytic solution.

As θ varies from 0 to π, y varies from 0 to 1. We now calculate $\dot{y}$

$$\dot{y} = \tfrac{1}{2}\left(\cos\frac{\theta}{2}\right)\dot{\theta} = \tfrac{1}{2}(1-y^2)^{1/2}\dot{\theta}$$

where we have used the substitution $\cos\theta/2 = (1-y^2)^{1/2}$

We now solve for $\dot{\theta}$ in terms of y and $\dot{y}$

$$\dot{\theta} = \frac{2\dot{y}}{(1-y^2)^{1/2}} = 2\left(\frac{mgl}{I}\right)^{1/2}(1-y^2)^{1/2}$$

We can now find a first-order differential equation describing the motion in terms of y

$$\dot{y} = \left(\frac{mgl}{I}\right)^{1/2}(1-y^2)$$

The solution is

$$y = \tanh\left(\frac{mgl}{I}\right)^{1/2} t$$

(b) As $t \to \infty$, $y \to 1$, and $\theta \to \pi$ and the pendulum goes "upside-down"—eventually. Compare this result with the last line in Table 8.4.1.

8.5 The Angular Momentum of a Rigid Body in Laminar Motion

Laminar motion takes place when all the particles that make up a rigid body move parallel to some fixed plane. In general, the rigid body undergoes both translational and rotational acceleration. The rotation takes place about an axis whose direction, but not necessarily its location, remains fixed in space. The rotation of a rigid body about a fixed axis is a special case of laminar motion, such as the physical pendulum discussed in the previous section. A cylinder rolling down an inclined plane is another example. We discuss motion of each of these types in the sections that follow, but as a prelude to these analyses, we first develop a theorem about the angular momentum of a rigid body in laminar motion.

We showed in Section 7.2 that the rate of change of the angular momentum of any system of particles is equal to the net applied torque

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{N} \tag{8.5.1}$$

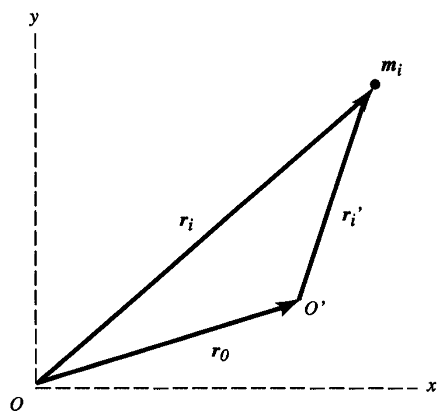


Figure 8.5.1 Vector position of a particle in a rigid body in laminar motion.

or

$$\frac{d}{dt}\sum_i \mathbf{r}_i \times m_i\mathbf{v}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i \tag{8.5.2}$$

where all quantities are referred to an inertial coordinate system.

What happens, however, if we choose to describe the rotation of a rigid body (which is a system of particles whose relative positions are fixed) about an axis that might also be accelerating, such as that which takes place when a ball rolls down an inclined plane? To take into account such a possibility, we again consider a system of particles, as in Section 8.2, that is rotating about an axis whose direction is fixed in space. However, here we allow for the possibility that the axis might be accelerating. We begin by referring the position of a particle, m_i, to the origin, O, of an inertial frame of reference (see Figure 8.5.1). Let the point O' represent the origin of the axis in question, about which we wish to refer the rotation of the system of particles. The vectors, $\mathbf{r}_i$ and $\mathbf{r}_i'$, denote the position of the ith particle relative to the points O and O', respectively. We now calculate the total torque $\mathbf{N}'$ about the axis O'

$$\mathbf{N}' = \sum_i \mathbf{r}_i' \times \mathbf{F}_i \tag{8.5.3}$$

From Figure 8.5.1, we see that

$$\mathbf{r}_i = \mathbf{r}_0 + \mathbf{r}_i' \tag{8.5.4}$$

and

$$\mathbf{v}_i = \mathbf{v}_0 + \mathbf{v}_i' \tag{8.5.5}$$

In the inertial frame of reference we have

$$\mathbf{F}_i = \frac{d}{dt}(m\mathbf{v}_i) \tag{8.5.6}$$

Thus, Equation 8.5.4 becomes

$$\mathbf{N}' = \sum_i \mathbf{r}_i' \times \mathbf{F}_i = \sum_i \mathbf{r}_i' \times \frac{d}{dt} m_i(\mathbf{v}_0 + \mathbf{v}_i') \tag{8.5.7a}$$

$$= -\dot{\mathbf{v}}_0 \times \sum_i m_i \mathbf{r}_i' + \sum_i \mathbf{r}_i' \times \frac{d}{dt} m_i \mathbf{v}_i' \tag{8.5.7b}$$

$$= -\dot{\mathbf{v}}_0 \times \sum_i m_i \mathbf{r}_i' + \frac{d}{dt} \sum_i \mathbf{r}_i' \times m_i \mathbf{v}_i' \tag{8.5.7c}$$

The step from Equation 8.5.7a to 8.5.7b follows because $\dot{\mathbf{v}}_0$ is not being summed and, therefore, may be extracted from the summation with impunity. The minus sign emerges because of the reversal of the order of the cross product. Extraction of the time derivative from inside the summation in Equation 8.5.7b to its position outside the summation in Equation 8.5.7c is permissible because it then generates a term, $\Sigma_i \mathbf{v}_i' \times m_i \mathbf{v}_i'$, that is the cross product of a vector with itself, which is zero.

The last term on the right in Equation 8.5.7c is the rate of change of the angular momentum, $\mathbf{L}'$, about the O' axis. Thus, we may rewrite this equation as

$$\mathbf{N}' = -\ddot{\mathbf{r}}_0 \times \sum_i m_i \mathbf{r}_i' + \frac{d}{dt} \mathbf{L}' \tag{8.5.8}$$

in which we have replaced $\dot{\mathbf{v}}_0$ with $\ddot{\mathbf{r}}_0$.

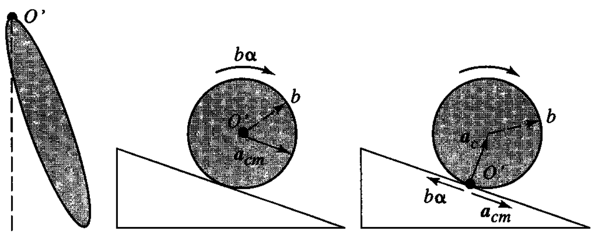
The equation of torque (8.5.1), thus, cannot be applied directly in its standard form to a system rotating about an axis that is undergoing acceleration. The correct equation (8.5.8) differs from Equation 8.5.1 by the presence of the extra term on the left.

However, this added term vanishes when any of three possible conditions are satisfied, as schematized in Figure 8.5.2a, b, and c:

1. The acceleration, $\ddot{\mathbf{r}}_0$, of the axis of rotation, O', vanishes (Figure 8.5.2a).
2. The point, O', is the center of mass of the system of particles that make up the rigid body. Under this condition, the term, $\Sigma_i m_i \mathbf{r}_i' = 0$ by definition (Figure 8.5.2b).
3. The O' axis passes through the point of contact between the cylinder and the plane. The vector, represented by the sum, $\Sigma_i m_i \mathbf{r}_i'$, passes through the center of mass. We can see this by noting that $\Sigma_i m_i \mathbf{r}_i' = M\mathbf{r}_{cm}'$ where $M = \Sigma_i m_i$ is the total mass and $\mathbf{r}_{cm}'$ is the vector position of the center of mass relative to O'. Therefore, *if* the vector $\ddot{\mathbf{r}}_0$ also passes through the center of mass, then their cross product will vanish (Figure 8.5.2c)

We will see in the next section that this last condition proves useful when solving problems involving rigid bodies that are rolling, but not sliding!

Condition 2 above should be emphasized. *The equation of torque for a rigid body undergoing laminar motion can always be expressed in the form given by Equation 8.5.1, if we take torques and calculate angular momentum about an axis that passes through the center of mass.* We write the equation here using appropriate notation to emphasize that it must be applied by summing torques about an axis that passes through the center of



(a) (b) (c)

Figure 8.5.2 (a) A physical pendulum swinging about a fixed axis O'. The acceleration of the axis $\ddot{\mathbf{r}}_0$ is zero. (b) A cylinder rolling down an inclined plane. An axis O' through its center of mass is accelerating, but Equation 8.5.9 may be used to describe its rotational motion. (c) The same cylinder as in (b) but the axis O' through the point of contact between the cylinder and the plane is accelerating, even though it is instantaneously at rest (no slipping). (The net tangential acceleration of the axis is zero because $\boldsymbol{a}_{cm} = b\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is the angular acceleration of the cylinder and b is its radius. The net acceleration of the axis is, therefore, its centripetal acceleration, $\boldsymbol{a}_C$, directed toward the center of mass.)

mass of the rigid body

$$\mathbf{N}_{cm} = \frac{d}{dt}\mathbf{L}_{cm} = I_{cm}\dot{\boldsymbol{\omega}} \tag{8.5.9}$$

If in doubt, use this equation!

8.6 | Examples of the Laminar Motion of a Rigid Body

To sum up, if a rigid body undergoes a laminar motion, the motion is most often specified as a translation of its center of mass and a rotation about an axis that passes through the center of mass and whose direction is fixed in space. Sometimes though, some other axis is a more appropriate choice. Such situations are usually obvious, as in the case of the physical pendulum, whose motion is a rotation about the fixed axis that passes through its pivot point.

The fundamental equation that governs the translation of a rigid body is

$$\mathbf{F} = m\ddot{\mathbf{r}}_{cm} = m\dot{\mathbf{v}}_{cm} = m\mathbf{a}_{cm} \tag{8.6.1}$$

where $\mathbf{F}$ is the vector sum of all the external forces acting on the body, m is its mass, and $\mathbf{a}_{cm}$ is the acceleration of its center of mass.

The fundamental equation that governs the rotation of the body about an axis O' that satisfies one of the conditions 1 to 3 given in Section 8.5 is

$$\mathbf{N}_{O'} = \frac{d}{dt}\mathbf{L}_{O'} = I_{O'}\,\dot{\boldsymbol{\omega}} \tag{8.6.2}$$

If an axis of rotation, other than that which passes through the center of mass, is chosen to describe the rotational motion, care should be taken in considering whether condition 1 or 3 is satisfied. If not, then the more general form of the equation of torque given by Equation 8.5.8 must be used instead.

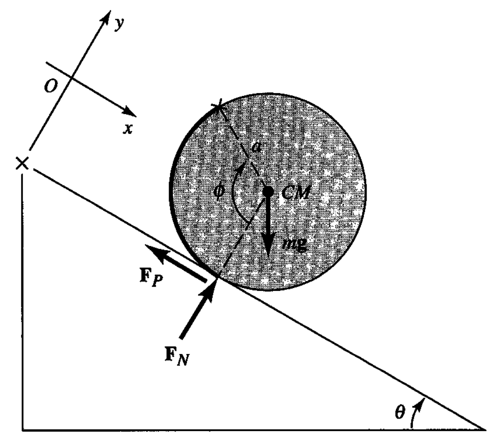


Figure 8.6.1 Body rolling down an inclined plane.

Body Rolling Down an Inclined Plane

As an illustration of laminar motion, we study the motion of a round object (cylinder, ball, and so on) rolling down an inclined plane. As shown in Figure 8.6.1, three forces are acting on the body. These are (1) the downward force of gravity, (2) the normal reaction of the plane $\mathbf{F}_N$, and (3) the frictional force parallel to the plane $\mathbf{F}_P$. Choosing axes as shown, the component equations of the translation of the center of mass are

$$m\ddot{x}_{cm} = mg \sin\theta - F_P \tag{8.6.3}$$

$$m\ddot{y}_{cm} = -mg \cos\theta + F_N \tag{8.6.4}$$

where θ is the inclination of the plane to the horizontal. Because the body remains in contact with the plane, we have

$$y_{cm} = \text{constant} \tag{8.6.5a}$$

Hence,

$$\ddot{y}_{cm} = 0 \tag{8.6.5b}$$

Therefore, from Equation 8.6.4,

$$F_N = mg \cos\theta \tag{8.6.6}$$

The only force that exerts a moment about the center of mass is the frictional force $\mathbf{F}_P$. The magnitude of this moment is $F_P a$, where a is the radius of the body. Hence, the rotational equation (Equation 8.6.2) becomes

$$I_{cm}\dot{\omega} = F_P a \tag{8.6.7}$$

To discuss the problem further, we need to make some assumptions regarding the contact between the plane and the body. We solve the equations of motion for two cases.

Motion with No Slipping

If the contact is very rough so that no slipping can occur, that is, if $F_P \le \mu_s F_N$, where μ_s is the coefficient of *static* friction, we have the following relations:

$$\dot{x}_{cm} = a\dot{\phi} = a\omega \tag{8.6.8a}$$

$$\ddot{x}_{cm} = a\ddot{\phi} = a\dot{\omega} \tag{8.6.8b}$$

where ϕ is the angle of rotation. Equation 8.6.7 can then be written

$$\frac{I_{cm}}{a^2}\ddot{x}_{cm} = F_P \tag{8.6.9}$$

Substituting this value for F_P into Equation 8.6.3 yields

$$m\ddot{x}_{cm} = mg\,\sin\theta - \frac{I_{cm}}{a^2}\ddot{x}_{cm} \tag{8.6.10}$$

Solving for $\ddot{x}_{cm}$, we find

$$\ddot{x}_{cm} = \frac{mg\,\sin\theta}{m + (I_{cm}/a^2)} = \frac{g\,\sin\theta}{1 + \left(k_{cm}^2/a^2\right)} \tag{8.6.11}$$

where k_{cm} is the radius of gyration about the center of mass. The body, therefore, rolls down the plane with constant linear acceleration and with constant angular acceleration by virtue of Equations 8.6.8a and b.

For example, the acceleration of a uniform cylinder $(k_{cm}^2 = a^2/2)$ is

$$\ddot{x}_{cm} = \frac{g\,\sin\theta}{1 + \left(\frac{1}{2}\right)} = \frac{2}{3}g\,\sin\theta \tag{8.6.12}$$

whereas that of a uniform sphere $(k_{cm}^2 = 2a^2/5)$ is

$$\ddot{x}_{cm} = \frac{g\,\sin\theta}{1 + \left(\frac{2}{5}\right)} = \frac{5}{7}g\,\sin\theta \tag{8.6.13}$$

EXAMPLE 8.6.1

Calculate the center of mass acceleration of the cylinder rolling down the inclined plane in Figure 8.6.1 for the case of no slipping. Choose an axis O' that passes through the point of contact as in Figure 8.5.2c.

Solution:

As previously explained, the choice of this axis satisfies condition 3 given in Section 8.5 and we can use Equation 8.6.2 directly. The torque acting about O' is

$$N_{O'} = mg\,a\,\sin\theta$$

The moment of inertia of the cylinder about this point (see Equation 8.3.22) is

$$I_{O'} = \tfrac{3}{2}ma^2$$

Because there is no slipping, the relationship between the angular velocity of the cylinder about the axis O' and the center of mass velocity is

$$\dot{x}_{cm} = a\dot{\phi}$$

(**Note:** this is the same relationship that connects the angular velocity of the cylinder with the tangential velocity of any point on its surface *relative to the center of mass*.)

Therefore, the rotational equation of motion gives

$$mga\ \sin\theta = \tfrac{3}{2}ma^2\left(\frac{\ddot{x}_{cm}}{a}\right)$$

from which it immediately follows that

$$\ddot{x}_{cm} = \tfrac{2}{3}g\ \sin\theta$$

Energy Considerations

The preceding results can also be obtained from energy considerations. In a uniform gravitational field the potential energy V of a rigid body is given by the sum of the potential energies of the individual particles, namely,

$$V = \sum_i (m_i g h_i) = mgh_{cm} \tag{8.6.14}$$

where h_{cm} is the vertical distance of the center of mass from some (arbitrary) reference plane. Now if the forces, other than gravity, acting on the body do no work, then the motion is conservative, and we can write

$$T + V = T + mgh_{cm} = E = \text{constant} \tag{8.6.15}$$

where T is the kinetic energy.

In the case of the body rolling down the inclined plane (see Figure 8.6.1), the kinetic energy of translation is $\frac{1}{2}m\dot{x}_{cm}^2$ and that of rotation is $\frac{1}{2}I_{cm}\omega^2$, so the energy equation reads

$$\tfrac{1}{2}m\dot{x}_{cm}^2 + \tfrac{1}{2}I_{cm}\omega^2 + mgh_{cm} = E \tag{8.6.16}$$

But $\omega = \dot{x}_{cm}/a$ and $h_{cm} = -x_{cm}\sin\theta$. Hence,

$$\tfrac{1}{2}m\dot{x}_{cm}^2 + \tfrac{1}{2}mk_{cm}^2\frac{\dot{x}_{cm}^2}{a^2} - mgx_{cm}\sin\theta = E \tag{8.6.17}$$

In the case of pure rolling motion, the frictional force does not appear in the energy equation because no mechanical energy is converted into heat unless slipping occurs. Thus, the total energy E is constant. Differentiating with respect to t and collecting terms yields

$$m\dot{x}_{cm}\ddot{x}_{cm}\left(1 + \frac{k_{cm}^2}{a^2}\right) - mg\dot{x}_{cm}\sin\theta = 0 \tag{8.6.18}$$

Canceling the common factor $\dot{x}_{cm}$ (assuming, of course, that $\dot{x}_{cm} \neq 0$) and solving for $\ddot{x}_{cm}$, we find the same result as that obtained previously using forces and moments (Equation 8.6.11).

Occurrence of Slipping

Let us now consider the case in which the contact with the plane is not perfectly rough but has a certain coefficient of *sliding* friction μ_k. If slipping occurs, then the magnitude of the frictional force $\mathbf{F}_P$ is given by

$$F_P = \mu_k F_N = \mu_k mg \cos\theta \tag{8.6.19}$$

The equation of translation (Equation 8.6.3) then becomes

$$m\ddot{x}_{cm} = mg \sin\theta - \mu_k mg \cos\theta \tag{8.6.20}$$

and the rotational equation (Equation 8.6.7) is

$$I_{cm}\dot{\omega} = \mu_k mga \cos\theta \tag{8.6.21}$$

From Equation 8.6.20 we see that again the center of mass undergoes constant acceleration:

$$\ddot{x}_{cm} = g(\sin\theta - \mu_k \cos\theta) \tag{8.6.22}$$

and, at the same time, the angular acceleration is constant:

$$\dot{\omega} = \frac{\mu_k mga \cos\theta}{I_{cm}} = \frac{\mu_k ga \cos\theta}{k_{cm}^2} \tag{8.6.23}$$

Let us integrate these two equations with respect to t, assuming that the body starts from rest, that is, at $t = 0$, $\dot{x}_{cm} = 0, \dot{\phi} = 0$. We obtain

$$\dot{x}_{cm} = g(\sin\theta - \mu_k \cos\theta)t \tag{8.6.24}$$

$$\omega = \dot{\phi} = g\left(\frac{\mu_k a \cos\theta}{k_{cm}^2}\right)t \tag{8.6.25}$$

Consequently, the linear speed and the angular speed have a constant ratio, and we can write

$$\dot{x}_{cm} = \gamma a\omega \tag{8.6.26}$$

where

$$\gamma = \frac{\sin\theta - \mu_k \cos\theta}{\mu_k a^2 \cos\theta / k_{cm}^2} = \frac{k_{cm}^2}{a^2}\left(\frac{\tan\theta}{\mu_k} - 1\right) \tag{8.6.27}$$

Now $a\omega$ cannot be greater than $\dot{x}_{cm}$, so γ cannot be less than unity. The limiting case, that for which we have pure rolling, is given by $\dot{x}_{cm} = a\omega$, that is,

$$\gamma = 1$$

Solving for μ_k in Equation 8.6.27 with $\gamma = 1$, we find that the critical value of the coefficient of friction is given by

$$\mu_{crit} = \frac{\tan\theta}{1+(a/k_{cm})^2} \tag{8.6.28}$$

(Actually this is the critical value for the coefficient of *static* friction μ_s.) If μ_s is greater than that given in Equation 8.6.28, then the body rolls without slipping.

For example, if a ball is placed on a 45° plane, it will roll without slipping, provided μ_s is greater than tan 45°/(1 + $\frac{5}{2}$) or $\frac{2}{7}$.

EXAMPLE 8.6.2

A small, uniform cylinder of radius R rolls without slipping along the inside of a large, fixed cylinder of radius $r > R$ as shown in Figure 8.6.2. Show that the period of small oscillations of the rolling cylinder is equivalent to that of a simple pendulum whose length is $3(r - R)/2$.

Solution:

A key to an easy solution hinges on the realization that the total energy of the rolling cylinder is a constant of the motion. There is no relative motion between the two surfaces because there is no slipping. In other words, O' and O coincide when the small cylinder is at the equilibrium position and the arc lengths $O'P$ and OP are identical. The force of friction **F**, therefore, does not remove the energy from the rolling cylinder, nor does the normal force **N** do any work. It generates no torque because its line of action always passes through the center of mass, and it does not affect the translational kinetic energy

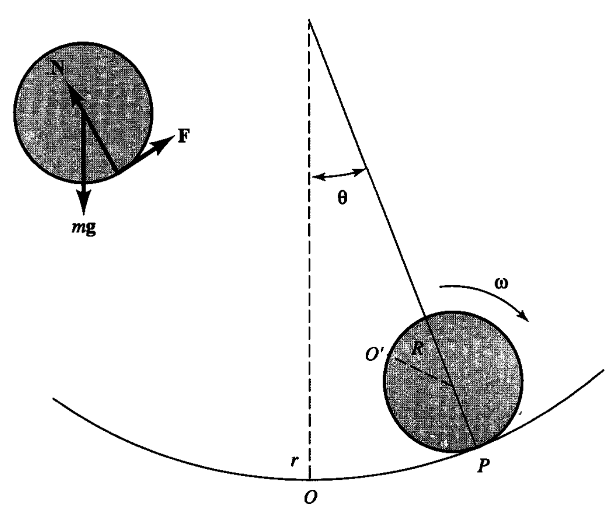


Figure 8.6.2 Small cylinder rolling without slipping on the inside of a large, fixed cylinder.

because it is always directed perpendicular to the motion of the center of the mass. The only force that does do work is the conservative force of gravity, $m\mathbf{g}$. Thus, the energy of the cylinder is conserved, and we can solve the problem by setting its time derivative equal to zero. The total energy of the cylinder is

$$E = T + V = \tfrac{1}{2} I_{cm}\omega^2 + \tfrac{1}{2} m v_{cm}^2 + mgh$$

where h is the height of the cylinder above that at its equilibrium position, v_{cm} is the speed of its center of mass, and I_{cm} is the moment of inertia about its center of mass (see Figure 8.6.2).

From the figure, we see that for small oscillations

$$h = (r - R)(1 - \cos\theta) \approx \tfrac{1}{2}(r - R)\theta^2$$

and because the cylinder rolls without slipping, we have

$$\omega = \frac{v_{cm}}{R} = \frac{(r - R)}{R}\dot{\theta}$$

Inserting these relations for h and ω into the energy equation gives

$$E = \frac{I_{cm}}{2R^2}(r - R)^2\dot{\theta}^2 + \frac{m}{2}(r - R)^2\dot{\theta}^2 + \frac{m}{2}g(r - R)\theta^2$$

On taking the derivative of the preceding equation and setting the result equal to zero, we obtain

$$\dot{E} = \frac{I_{cm}}{R^2}(r - R)^2\ddot{\theta}\dot{\theta} + m(r - R)^2\ddot{\theta}\dot{\theta} + mg(r - R)\theta\dot{\theta} = 0$$

and cancelling out common terms yields

$$\left(\frac{I_{cm}}{R^2} + m\right)(r - R)\ddot{\theta} + mg\theta = 0$$

The moment of inertia of the cylinder about its center of mass is $I_{cm} = mR^2/2$, and on substituting it into the preceding equation yields the equation of motion of the cylinder for small excursions about equilibrium

$$\ddot{\theta} + \frac{g}{\left(\frac{3}{2}\right)(r - R)}\theta = 0$$

This equation of motion is the same as that of a simple pendulum of length $3(r - R)/2$. Thus, their periods are identical.

(The student might wish to solve this problem using the method of forces and torques. The relevant forces acting on the rolling cylinder are shown in the insert in Figure 8.6.2.)

8.7 Impulse and Collisions Involving Rigid Bodies

In the previous chapter we considered the case of an impulsive force acting on a particle. In this section we extend the notion of impulsive force to the case of laminar motion of a rigid body. First, we know that the translation of the body, assuming constant mass, is governed by the general equation $\mathbf{F} = m\, d\mathbf{v}_{cm}/dt$, so that if $\mathbf{F}$ is an impulsive type of force, the change of linear momentum of the body is given by

$$\int \mathbf{F}\, dt = \boldsymbol{P} = m\Delta\mathbf{v}_{cm} \tag{8.7.1}$$

Thus, the result of an impulse $\boldsymbol{P}$ is to produce a sudden change in the velocity of the center of mass by an amount

$$\Delta\mathbf{v}_{cm} = \frac{\boldsymbol{P}}{m} \tag{8.7.2}$$

Second, the rotational part of the motion of the body obeys the equation $N = \dot{L} = Id\omega/dt$, so the change in angular momentum is

$$\int N dt = I\Delta\omega \tag{8.7.3}$$

The integral $\int N\, dt$ is called the *rotational impulse.* Now if the primary impulse $\boldsymbol{P}$ is applied to the body in such a way that its line of action is a distance l from the reference axis about which the angular momentum is calculated, then $N = Fl$, and we have

$$\int N dt = Pl \tag{8.7.4}$$

Consequently, the change in angular velocity produced by an impulse $\boldsymbol{P}$ acting on a rigid body in laminar motion is given by

$$\Delta\omega = \frac{Pl}{I} \tag{8.7.5}$$

For the general case of free laminar motion, the reference axis must be taken through the center of mass, and the moment of inertia $I = I_{cm}$. On the other hand, if the body is constrained to rotate about a fixed axis, then the rotational equation alone suffices to determine the motion, and I is the moment of inertia about the fixed axis.

In collisions involving rigid bodies, the forces and, therefore, the impulses that the bodies exert on one another during the collision are always equal and opposite. Thus, the principles of conservation of linear and angular momentum apply.

Center of Percussion: The "Baseball Bat Theorem"

To illustrate the concept of center of percussion, let us discuss the collision of a ball of mass m, treated as a particle, with a rigid body (bat) of mass M. For simplicity we assume that the body is initially at rest on a smooth horizontal surface and is free to move in laminar-type motion. Let $\boldsymbol{P}$ denote the impulse delivered to the body by the ball. Then the equations for translation are

$$\boldsymbol{P} = M\mathbf{v}_{cm} \tag{8.7.6}$$

$$-\boldsymbol{P} = m\mathbf{v}_1 - m\mathbf{v}_0 \tag{8.7.7}$$

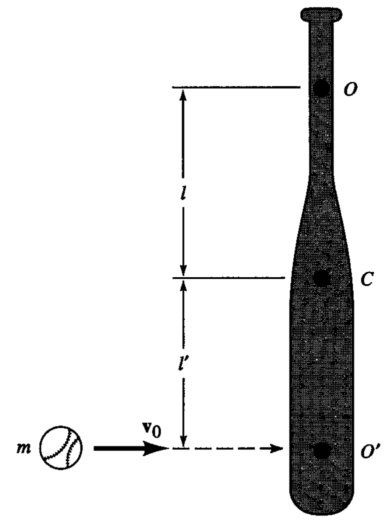


Figure 8.7.1 Baseball colliding with a bat.

where $\mathbf{v}_0$ and $\mathbf{v}_1$ are, respectively, the initial and final velocities of the ball and $\mathbf{v}_{cm}$ is the velocity of the mass center of the body after the impact. The preceding two equations imply conservation of linear momentum.

Because the body is initially at rest, the rotation about the center of mass, as a result of the impact, is given by

$$\omega = \frac{Pl'}{I_{cm}} \tag{8.7.8}$$

in which l' is the distance $O'C$ from the center of mass C to the line of action of $\boldsymbol{P}$, as shown in Figure 8.7.1. Let us now consider a point O located a distance l from the center of mass such that the line CO is the extension of $O'C$, as shown. The (scalar) velocity of O is obtained by combining the translational and rotational parts, namely,

$$v_O = v_{cm} - \omega l = \frac{P}{M} - \frac{Pl'}{I_{cm}} l = P\left(\frac{1}{M} - \frac{ll'}{I_{cm}}\right) \tag{8.7.9}$$

In particular, the velocity of O will be zero if the quantity in parentheses vanishes, that is, if

$$ll' = \frac{I_{cm}}{M} = k_{cm}^2 \tag{8.7.10}$$

where k_{cm} is the radius of gyration of the body about its center of mass. In this case the point O is the instantaneous center of rotation of the body just after impact. O' is called the *center of percussion* about O. The two points are related in the same way as the centers of oscillation, defined previously in our analysis of the physical pendulum (Equation 8.4.13).

Anyone who has played baseball knows that if the ball hits the bat in just the right spot there is no "sting" on impact. This "right spot" is just the center of percussion about the point at which the bat is held.

EXAMPLE 8.7.1

Shown in Figure 8.7.2 is a thin rod of length b and mass m suspended from an endpoint on a frictionless pivot. The other end of the rod is struck a blow that delivers a horizontal impulse P' to the rod. Calculate the horizontal impulse P delivered to the pivot by the suspended rod.

Solution:

First, we calculate the velocity of the center of mass after the blow by noting that the net horizontal impulse delivered to the rod is equal to its change in momentum.

$$P' - P = mv_{cm}$$

Now we consider the resulting rotation of the rod about the pivot point (the choice of this axis satisfies condition 1 in Section 8.5). The moment of inertia of the rod about an axis passing through that point is given by Equation 8.3.3

$$I = \tfrac{1}{3}mb^2$$

Now we calculate the angular velocity of the rod about the pivot using Equation 8.7.8

$$P'b = I\omega$$

But the velocity of the center of mass and the angular velocity are related according to

$$v_{cm} = \frac{b}{2}\omega$$

Thus, we can write

$$P' - P = m\frac{b}{2}\omega = m\frac{b}{2}\left[\frac{P'b}{I}\right] = \tfrac{3}{2}P'$$

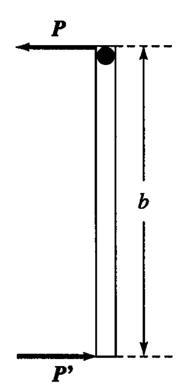


Figure 8.7.2 Thin rod suspended from frictionless pivot.

Therefore,

$$P = -\tfrac{1}{2}P'$$

The impulse delivered by the pivot to the rod is in the same direction as the impulse delivered by the horizontal blow, to the right in Figure 8.7.2. The impulse delivered by the rod to the pivot is in the opposite direction, to the left in the figure.

Problems

8.1 Find the center of mass of each of the following:
- **(a)** A thin wire bent into the form of a three-sided, block-shaped "⊔" with each segment of equal length b
- **(b)** A quadrant of a uniform circular lamina of radius b
- **(c)** The area bounded by parabola $y = x^2/b$ and the line $y = b$
- **(d)** The volume bounded by paraboloid of revolution $z = (x^2 + y^2)/b$ and the plane $z = b$
- **(e)** A solid uniform right circular cone of height b

8.2 The linear density of a thin rod is given by $\rho = cx$, where c is a constant and x is the distance measured from one end. If the rod is of length b, find the center of mass.

8.3 A solid uniform sphere of radius a has a spherical cavity of radius $a/2$ centered at a point $a/2$ from the center of the sphere. Find the center of mass.

8.4 Find the moments of inertia of each of the objects in Problem 8.1 about their symmetry axes.

8.5 Find the moment of inertia of the sphere in Problem 8.3 about an axis passing through the center of the sphere and the center of the cavity.

8.6 Show that the moment of inertia of a solid uniform octant of a sphere of radius a is $(\frac{2}{5})ma^2$ about an axis along one of the straight edges. (*Note:* This is the same formula as that for a solid sphere of the same radius.)

8.7 Show that the moments of inertia of a solid uniform rectangular parallelepiped, elliptic cylinder, and ellipsoid are, respectively, $(m/3)(a^2 + b^2)$, $(m/4)(a^2 + b^2)$, and $(m/5)(a^2 + b^2)$, where m is the mass, and $2a$ and $2b$ are the principal diameters of the solid at right angles to the axis of rotation, the axis being through the center in each case.

8.8 Show that the period of a physical pendulum is equal to $2\pi(d/g)^{1/2}$, where d is the distance between the point of suspension O and the center of oscillation O'.

8.9
- **(a)** An idealized simple pendulum consists of a particle of mass M suspended by a thin massless rod of length a. Assume that an actual simple pendulum consists of a thin rod of mass m attached to a spherical bob of mass $M - m$. If the radius of the spherical bob is equal to b, and the length of the thin rod is equal to $a - b$, calculate the ratio of the period of the actual simple pendulum to the idealized simple one.
- **(b)** Calculate a value for this ratio if $m = 10$ g, $M = 1$ kg, $a = 1.27$ m, and $b = 5$ cm.

8.10 The period of a physical pendulum is 2 s. (Such a pendulum is called a "seconds" pendulum.) The mass of the pendulum is M, and its center of mass is 1 m below the axis of oscillation. A particle of mass m is attached to the bottom of the pendulum, 1.3 m below the axis, in line with the center of gravity. It is then found that the pendulum "loses" time at the rate of 20 s/day. Find the ratio of m to M.

8.11 A circular hoop of radius a swings as a physical pendulum about a point on the circumference. Find the period of oscillation for small amplitude if the axis of rotation is