Circular motion and the unit circle

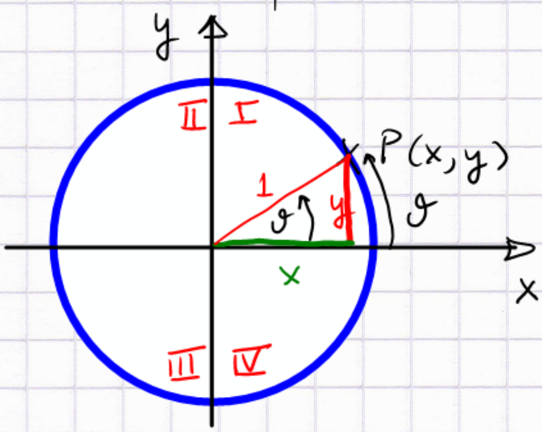
Motion on a circle: 2 dimensions $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}$

constrained by: $x(t)^2 + y(t)^2 = R^2$ ← radius

should be described using a <u>single</u> degree of freedom

= one real variable → polar angle $\vartheta(t)$

unit circle (mathematics)



ϑ = angle in degrees

ϑ = arclength in radians

conversion: $180^\circ \mathrel{\hat{=}} \pi$ radians

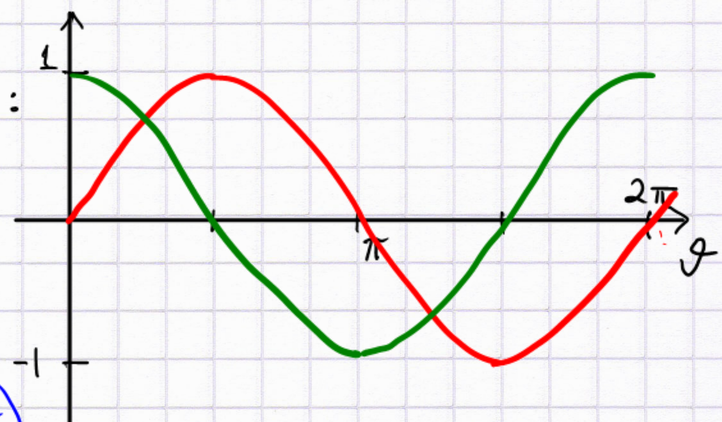
$\pi = 3.14(159265\ldots)$

$x = \cos\vartheta$ $\qquad$ $y = \sin\vartheta$

As ϑ varies from $0 \ldots 2\pi$:

We need sin, cos and $\tan = \frac{\sin}{\cos}$

in many contexts:



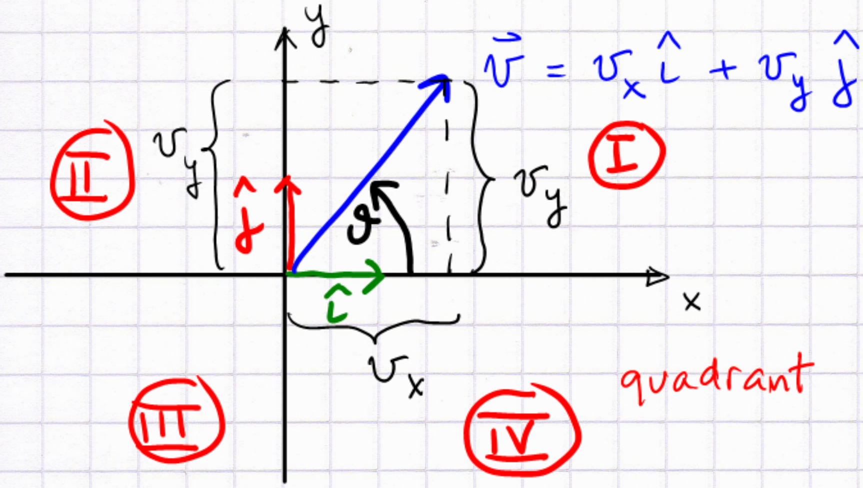
<u>Example 1:</u> $\sin(\omega t) = \sin(2\pi f t)$

= oscillatory (in time) motion, electrical signal

<u>Example 2:</u> Vector $\vec{v} = v_x\,\hat{\imath} + v_y\,\hat{\jmath}$ to be expressed

as length & <u>direction</u>. $v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$

Direction: use ϑ = angle from positive x-axis to vector, in pos. math. sense (= counterclockwise CCW).



Inverse trig functions

$$\frac{v_y}{v_x} = \tan\vartheta$$

$$\tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}(\tan\vartheta) = \vartheta$$

Note: By default $\tan^{-1}$ returns values in the range: $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$ (quadrants I, IV)

$\frac{v_y}{v_x} > 0$: $\Rightarrow$ I $\qquad$ $\frac{v_y}{v_x} < 0$: IV

Suppose the vector points into quadrant II $\quad$ $v_x < 0$, $v_y > 0$

$\tan^{-1}$ receives a neg. argument $\frac{v_y}{v_x}$, answers: ϑ in IV

How do we fix this? $\quad$ Need to add π to ϑ

Alternative: use $v = \sqrt{v_x^2 + v_y^2}$, $\vartheta = \cos^{-1}\left(\frac{v_x}{v}\right)$?

This works in I, II, since $\cos^{-1}$ maps the interval $(-1, 1)$ into $(\pi, 0)$, i.e., $0 < \vartheta < \pi$ is the range

So, try $\vartheta = \sin^{-1}\left(\frac{v_y}{v}\right)$? $\quad$ always yields $-\frac{\pi}{2} < \vartheta < \frac{\pi}{2}$

Always make a sketch, use v_x, v_y sign info to figure out which quadrant the tip points to, adjust ϑ by adding π.

Aside

Small caveat: suppose $v_x = 1.5$, $v_y = -0.01$

$\therefore \; \vartheta = \tan^{-1}(-0.015) = \tan^{-1}(-1.5 \times 10^{-2}) = -1.5 \times 10^{-2}$

$\xrightarrow{\text{convert to degrees}} \; \vartheta = -0.86^\circ$ (2 significant digits)

Express as a positive angle (quadrant IV: $270^\circ < \vartheta < 360^\circ$)

$360^\circ - 0.86^\circ = 359.14^\circ$ ← are we quoting 5 significant digits here? NOT REALLY!

$\therefore$ Advantage in using negative angles when $|\vartheta|$ is small.

Uniform circular motion on circle of radius R:

$$\vec{r}(t) = R\left[\cos\vartheta(t)\,\hat{\imath} + \sin\vartheta(t)\,\hat{\jmath}\right]$$

when $t=0$ implies $\vartheta = 0$, the motion starts on the positive x-axis: $\vec{r}(0) = R\,\hat{\imath}$

Uniform motion is characterized by a period T (it takes T seconds to go once around)

The frequency $f = \frac{1}{T}$ measures revolutions / time

$$\vartheta(t) = 2\pi \frac{t}{T}$$

why? $0 < t < T$ is one orbit

$0 < \vartheta < 2\pi$ is one math. period

$$= 2\pi f t$$

$$= \omega t$$

defines $\omega = 2\pi f = \frac{2\pi}{T}$ "circular" frequency