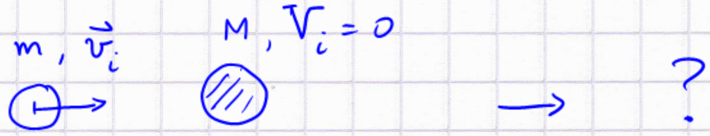
Linear Momentum Conservation

Motivation : 1) understand why an extended object made up of many mass segments (held together by pairwise forces) behaves under external force actions as one entity (point mass model)

2) understand 'billiard-ball' type collisions (elastic collisions)

3) understand explosions (break-up) and (inelastic) collisions with sticking

A) treat a system of N particles without external forces, only pairwise forces by Newton's 2^{nd} + 3^{rd} Laws.



without looking at the details (on the msec time scale)

except : $\{ \vec{F}_{m\,on\,M} , \vec{F}_{M\,on\,m} \}$ = force pair, i.e.,

$$\vec{F}_{m\,on\,M} = - \vec{F}_{M\,on\,m} \qquad \text{(Newton 3)}$$

$$\left.\begin{aligned} \Delta\vec{p}_m &= \vec{p}^{\,fin} - \vec{p}^{\,in} = \vec{F}_{M\,on\,m}\,\Delta t \\ \Delta\vec{P}_M &= \vec{P}^{\,fin} - \vec{P}^{\,in} = \vec{F}_{m\,on\,M}\,\Delta t \end{aligned}\right\} + \qquad \begin{matrix}\text{(momentum - impulse th.)} \\ \text{(= Newton 2)}\end{matrix}$$

$= 0$ due to $\sim\sim$

$$\vec{p}^{\,fin} - \vec{p}^{\,in} + \vec{P}^{\,fin} - \vec{P}^{\,in} = 0$$

$$\therefore \quad \boxed{\vec{p}^{\,fin} + \vec{P}^{\,fin} = \vec{p}^{\,in} + \vec{P}^{\,in}}$$

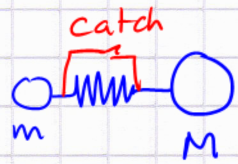
The total momentum for the 2-body system doesn't change [in the absence of external $\vec{F}$]

This statement generalizes to more than 2 particles

$$\sum_{i=1}^{N} \vec{p}_i^{\,in} = \sum_{i=1}^{N} \vec{p}_i^{\,fin}$$

in the absence of external forces!

Examples: 1) the relative force between m and M



Cannot change $\vec{P}_{tot} = m\vec{v} + M\vec{V}$

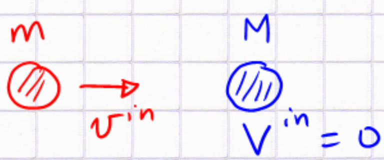
1d: release catch $\rightarrow$ $mv + MV = 0$

$$\therefore \quad v = -\frac{M}{m} V$$

small mass will be fast compared to big mass

2) billiard ball $\{m, v^{in}\}$ hits another $\{M, V^{in} = 0\}$

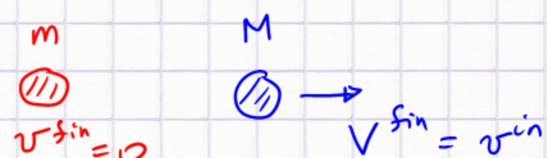
$M = m$



$P_{tot} = m v^{in}$

in 1d no vectors, but scalars (with sign)

$\longrightarrow$

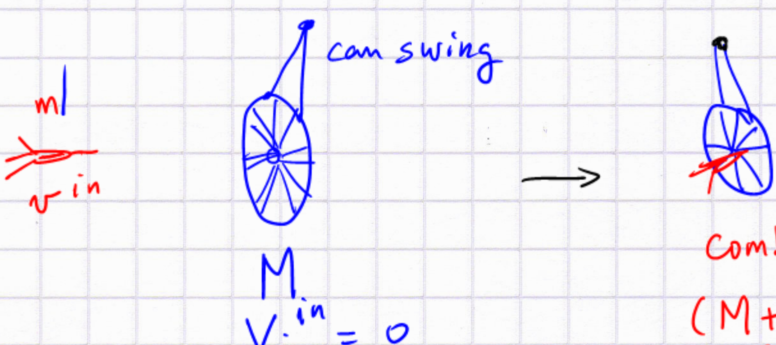


$(m = M!)$

observation is consistent with this, but it isn't explained yet!

3) arrow hits a suspended dart board

= ballistic pendulum $\rightarrow$ determine muzzle speed from shooting into suspended sandbox



combined object

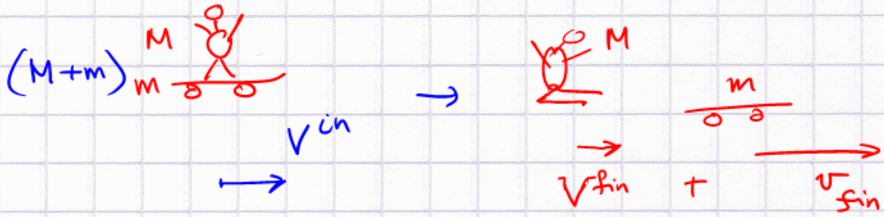
$$(M+m)\, V^{fin} = m\, v^{in}$$

$$V^{fin} = \frac{m}{m+M}\, v^{in}$$

Momentum conservation alone:

explains: sticky collisions (inelastic)

explosions; skate/snow boarder jumping off:



(no external forces during separation event)

$\vec{P}^{fin}_{tot} = \vec{P}^{in}_{tot}$!

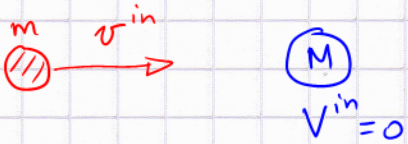
Energy Conservation

(elastic collisions only!)

Now figure out m colliding with M (1 dim.)

$$m v^{in} + M V^{in} = m v^{fin} + M V^{fin}$$

(total momentum is conserved when no external forces during collision)



$\rightarrow$?

e.g., for $M \gg m$.

m has initially KE^{in}_m: $\frac{1}{2} m (v^{in})^2$; $KE^{in}_M = 0$

Collision: $\sum KE$ converts into PE (deformation; spring like, then re-conversion to KE)

$$\frac{1}{2} m (v^{in})^2 + \frac{1}{2} M (V^{in})^2 = \frac{1}{2} m (v^{fin})^2 + \frac{1}{2} M (V^{fin})^2$$

Consistent with momentum reversal when ball hits a wall (wall doesn't move; $KE^{in}_m = KE^{fin}_m$) (but momentum conservation is weird!)

Do the case $V^{in} = 0$, $M > m$ but not infinite (wall) $\rightarrow$ can move! $V^{fin} \neq 0$

① Mom. cons: $m v^{in} = m v^{fin} + M V^{fin}$

② KE cons: $\frac{1}{2} m (v^{in})^2 = \frac{1}{2} m (v^{fin})^2 + \frac{1}{2} M (V^{fin})^2$

} 2 eqs in 2 unknowns $\{v^{fin}, V^{fin}\}$

use ① in ② to eliminate v^{fin}

$$v^{fin} = v^{in} - \frac{M}{m} V^{fin} \quad \leftarrow \text{use in } ②$$

$$\frac{1}{2} M (V^{fin})^2 + \underbrace{\frac{1}{2} m \left(v^{in} - \frac{M}{m} V^{fin} \right)^2} = \frac{1}{2} m (v^{in})^2$$

cancel

$$\frac{1}{2} m (v^{in})^2 - \frac{1}{2} m \frac{2M}{m} (v^{in}) V^{fin} + \frac{1}{2} m \left(\frac{M^2}{m^2}\right) (V^{fin})^2$$

$$- M (v^{in})(V^{fin}) + \frac{M^2}{2m} (V^{fin})^2$$

$$(V^{fin})^2 \left[\frac{M}{2} + \frac{M^2}{2m} \right] = M (v^{in})(V^{fin})$$

assume $V^{fin} \neq 0$ + divide by $M V^{fin}$

$$\frac{1}{2} V^{fin} \left[1 + \frac{M}{m} \right] = v^{in}$$

$$V^{fin} \left[\frac{m+M}{m} \right] = 2 v^{in}$$

$$\boxed{V^{fin} = \frac{2m}{m+M} v^{in}}$$

Discussion: 1) $m = M$ (billiards, Newton pendulum)

$$V^{fin} = v^{in} \quad ✓$$

2) $M \gg m$ $\qquad V^{fin} \to 0$

Q: and what happens to v^{fin} ?

$$v^{fin} = v^{in} - \frac{M}{m} V^{fin} = v^{in} - \frac{M}{\cancel{m}} \frac{2\cancel{m}}{m+M} v^{in}$$

$$= v^{in} \left(1 - \frac{2M}{m+M} \right) = v^{in} \left(\frac{m+M-2M}{m+M} \right)$$

$$= (v^{in}) \frac{m-M}{m+M} = (v^{in}) \frac{\frac{m}{M} - 1}{\frac{m}{M} + 1}$$

$$\xrightarrow[\left(\frac{m}{M}\right) \to 0]{} -(v^{in})$$

[limit momentum reversal in collision]

Now we understand: wall just needs to be heavy (could be sitting on ice!)

light particle bounces back without affecting M (much)