Angular momentum $\vec{L}$

(i) $\vec{L}$ of a pointlike particle defined by

$$\vec{L} = \vec{r} \times \vec{p}$$

e.g. for the ball attached to massless rod discussed above we have:

$$|\vec{L}| = |\vec{r}|\,|\vec{p}|\,|\sin\phi|$$

$$= r\,m\,\underbrace{v}_{=\omega r} \quad (\phi = \tfrac{\pi}{2} \text{ for circular motion})$$

$$= m r^2 \omega = I\omega$$

orientation : RH rule $\Rightarrow$ $\vec{L} = I\vec{\omega}$

Note that for uniform circular motion $\vec{\omega} = \text{const} \Rightarrow \vec{L} = \text{constant}$.

Now recall Newton-2 : $\frac{d}{dt}\vec{p} = \vec{F}$
(see discussion of impulse theorem)

Try something similar with $\vec{L}$:

$$\Leftarrow \quad \frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r} \times \vec{p}) \overset{\text{product rule}}{=} \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \underbrace{m\,\vec{v} \times \vec{v}}_{=0} + \vec{r} \times \vec{F} = \vec{\tau}$$

Summary : $\boxed{\frac{d}{dt}\vec{L} = \vec{\tau}}$ angular-momentum theorem

We can infer a conservation law:

if $\vec{\tau} = 0 \Rightarrow \vec{L}$ conserved

$\vec{\tau} = 0$ if $\begin{cases} \vec{F} = 0 & \text{(trivial)} \\ \vec{r} \parallel \vec{F} & \text{(e.g. uniform circular motion)} \end{cases}$

(ii) $\vec{L}$ for a particle system (extended object)

define $\vec{L}_{tot} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i (m_i \vec{r}_i \times \vec{v}_i)$

$$\vec{\tau}_{tot} = \sum_i \vec{r}_i \times \vec{F}_i$$

show (similarly as above): $\frac{d}{dt} \vec{L}_{tot} = \vec{\tau}_{tot}$ *)

For rotations about a fixed axis, one finds (again):

$$\vec{L}_{tot} = I \vec{\omega} \quad \text{with} \quad I = \sum_i m_i r_i^2$$

Examples for angular momentum conservation

(i) Spinning figure skater

$$\left.\begin{aligned} L_{initial} &\overset{\tau=0}{=} I_{initial}\,\omega_{initial} \\ &= L_{final} = I_{final}\,\omega_{final} \end{aligned}\right\} \Leftrightarrow \omega_{final} = \frac{I_{initial}}{I_{final}}\,\omega_{initial}$$

start with a small angular velocity $\omega_{initial}$, then pull arms and legs in to reduce I => end up with a higher angular velocity ω_{final}

Consider KE:

$$KE_f = \frac{1}{2} I_{final}\,\omega_{final}^2 = \frac{1}{2} I_{final}\left(\frac{I_{initial}}{I_{final}}\,\omega_{initial}\right)^2$$

$$= \left(\frac{I_{initial}}{I_{final}}\right)\frac{1}{2} I_{initial}\,\omega_{initial}^2 = \underbrace{\frac{I_{initial}}{I_{final}}}_{>1} KE_i$$

$$\Leftrightarrow \quad KE_f > KE_i$$

KE is <u>not</u> conserved, but increased (the skater does positive work when pulling in arms + legs)

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*) Angular momentum conservation, if $\vec{\tau}_{tot} = 0$:

$$\vec{L}_{tot} = \vec{L}_i = \vec{L}_f = \text{const}$$

(ii) Planetary motion

Recall: Earth orbits Sun on an elliptical orbit *)

$$\vec{F}_{S\,on\,E} = -\frac{GM_E M_S}{r^3}\,\vec{r}$$

Earth's position vector relative to resting sun

Q: does Sun exert a torque on Earth?

$$\vec{\tau} = \vec{r} \times \vec{F}_{S\,on\,E} = -\frac{GM_E M_S}{r^3}\,\underbrace{\vec{r}\times\vec{r}}_{=0} = 0$$

A: no $\Rightarrow$ $\vec{L}_{Earth} = \text{const}$

This has two important implications:

(i) Direction of $\vec{L}_{Earth}$ never changes $\Rightarrow$ orbit is truly planar

(ii) Magnitude $|\vec{L}_{Earth}|$ never changes; one can show that Kepler's area law follows from this fact

(see book, Chap 9.4)

So, Sun doesn't exert a torque on Earth. It does do work though. For Earth's (slightly) elliptical orbit $\vec{F}_{grav}$ is <u>not</u> always perpendicular to displacement (unlike for uniform circular motion).

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*) Assume Sun is infinitely heavy, i.e. $M_S \to \infty$ (such that it is at rest despite Newton-3.)