PHYS 6204 1.0: Elements of Quantum Scattering Theory (Winter 2016)

Assignment **Due: April 18, 6:00 p.m.**

1. (10 marks) Consider the asymptotic form of the stationary wave function for potential scattering
$$\psi(\mathbf{r}) \stackrel{r\to\infty}{\to} e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}. \tag{1}$$
The total current density associated with Eq. (1) can be cast into the form
$$\mathbf{j}(\mathbf{r}) = \mathbf{j}_{\text{inc}}(\mathbf{r}) + \mathbf{j}_{\text{sc}}(\mathbf{r}) + \mathbf{j}_{\text{int}}(\mathbf{r})$$
with
$$\begin{aligned} \mathbf{j}_{\text{inc}}(\mathbf{r}) &= \frac{\hbar k}{m}\hat{z} \\ \mathbf{j}_{\text{sc}}(\mathbf{r}) &= \frac{\hbar k|f(\theta,\phi)|^2}{mr^2}\hat{r}. \end{aligned}$$
Determine $\mathbf{j}_{\text{int}}(\mathbf{r})$ (at large r) and show that
$$\oiint \mathbf{j}_{\text{int}}(\mathbf{r})\cdot d\mathbf{S} = -\frac{4\pi\hbar}{m}\text{Im}f(\theta=0).$$
Use this result to prove the optical theorem
$$\sigma_{\text{tot}} = \frac{4\pi}{k}\text{Im}f(\theta=0).$$

2. (10 marks) Calculate the electrostatic potential
$$\Phi(r) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'$$
for the charge density associated with a hydrogen atom in the ground state[1]. Show that
$$\Phi(r) \stackrel{r\to 0}{\to} \frac{1}{4\pi\epsilon_0 r}.$$
Discuss the usefulness and the limitations of the potential energy function
$$V(r) = -e\Phi(r) \tag{2}$$
to describe elastic electron-hydrogen scattering.

[1] You can use atomic units if you prefer.

3. (20 marks) Consider the spherical potential well

$$V = \begin{cases} -V_0 & r < R \\ \\ 0 & r > R \end{cases} \tag{3}$$

with $V_0 > 0$.

(a) For S-wave scattering one can show that

$$\tan \delta_0 \overset{k \to 0}{\to} -ka \quad \text{with} \quad a = \frac{RL(R) - 1}{L(R)},$$

where $L(R) = \frac{K \cos KR}{\sin KR}$ with $K = \sqrt{k^2 + \frac{2m}{\hbar^2} V_0}$, and δ_0 and a being the S-wave scattering phase shift and scattering length, respectively. Show that for $kR \ll 1$ and $KR \ll 1$ one obtains the scattering amplitude

$$f(\theta) \approx -a \approx \frac{K^2 R^3}{3} \approx \frac{2mV_0 R^3}{3\hbar^2}$$

and the total cross section

$$\sigma_{\text{tot}} \approx \frac{16\pi}{9} \frac{m^2 V_0^2 R^6}{\hbar^4}. \tag{4}$$

(b) In the Born approximation one has

$$f(\theta) = -\frac{m}{2\pi\hbar^2} \tilde{V}(\mathbf{q}) \quad \text{with} \quad \tilde{V}(\mathbf{q}) = \int V(r) e^{-i\mathbf{q}\cdot\mathbf{r}} d^3 r,$$

where $\mathbf{q} = \mathbf{k}_f - \mathbf{k}_i$ is the momentum transfer, whose magnitude is related to the scattering angle by $q = 2k \sin \frac{\theta}{2}$ ($k = |\mathbf{k}_i| = |\mathbf{k}_f|$). Show for the potential well (3) by direct integration using spherical coordinates[2] that

$$\tilde{V}(\mathbf{q}) = \tilde{V}(q) = \frac{4\pi}{q} \int_0^\infty r V(r) \sin qr dr = -\frac{4\pi V_0}{q} \left(\frac{\sin qR}{q^2} - \frac{R \cos qR}{q} \right).$$

Reobtain (4) for $qR \ll 1$.

(c) (Bonus) Along similar lines, calculate the Born scattering amplitude for the potential (2).

4. (10 marks) Consider elastic scattering from a central potential in the Born approximation. We have seen in class that the T-matrix can be written in the form

$$T_{fi}^{B1} = \frac{\alpha}{q^2} \int \rho(r) e^{-i\mathbf{q}\cdot\mathbf{r}} d^3 r, \tag{5}$$

where α is a constant and $\rho(r)$ the radially symmetric charge density of the target.

(a) Assume a small momentum transfer such that the exponential in Eq. (5) can be Taylor-expanded. Find and interpret the lowest-order contribution to the T-matrix (5).

(b) Find and interpret the next contributing term to the T-matrix (5).

[2] You may find the integral $\int x \sin ax dx = (\sin ax)/(a^2) - (x \cos ax)/(a)$ useful.