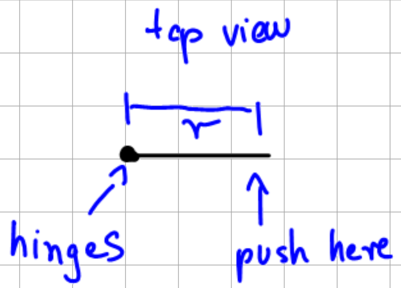
Torque

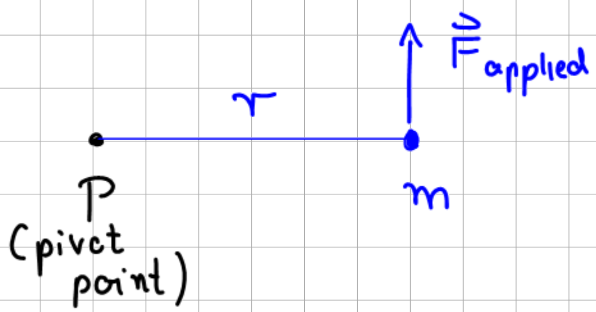
How do you open a door ?



you push it as far away from the hinges as possible. It's not just the applied force that matters, but the product of force and "lever arm", i.e. torque :

$$\tau = F_{applied}\, r$$

Another example : ball attached to (massless) rod



define torque as above :

$$\tau = F_{applied}\, r$$

Now study (nonuniform circular) motion of ball :

recap :
$$\vec{a}(t) = \vec{a}_c(t) + \vec{a}_t(t)$$
$$(\vec{a}_c \perp \vec{a}_t)$$

with $a_c = \omega^2 r$, $\omega = \dot{\theta}$

$a_t = \alpha r$, $\alpha = \dot{\omega} = \ddot{\theta}$ (note that a_t can be negative !)

Newton-2 :
$$m\vec{a} = \sum \vec{F}$$

$$\Leftrightarrow \begin{cases} m a_c = m\omega^2 r = T \\ m a_t = F_{applied} \end{cases}$$

(tension of rod)

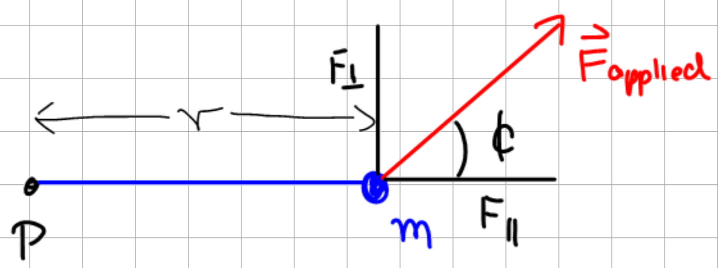
elaborate on 2nd equation

$$\Leftrightarrow \quad m\alpha r = F_{applied} \quad | \cdot r$$

$$\Leftrightarrow \quad \boxed{I\alpha = \tau}$$ Newton-like equation for rotational motion

with "moment of inertia" $I = mr^2$

Consider a variant of the example to generalize things:



$F_\perp = F_{applied} \sin\phi$

$F_\parallel = F_{applied} \cos\phi$

Newton-2 for this case:

$$m a_c = m\omega^2 r = T - F_\parallel$$

$$m a_t = F_\perp$$

$$\Leftrightarrow \quad m\alpha r = F_{applied} \sin\phi \quad | \cdot r$$

$$\Leftrightarrow \quad I\alpha = F_{applied}\, r \sin\phi =: \tau$$

interpretation ①. $\tau = F_\perp r$ with $F_\perp = F_{applied} \sin\phi$

interpretation ②. $\tau = F_{applied}\, r_\perp$ with $r_\perp = r \sin\phi$

(can you sketch this?)

So, we have a more general definition of (the magnitude of) torque if force and lever arm are not perpendicular.

Now make it really general by elevating torque to a vector:

(i) define magnitude of $\vec{\tau}$:

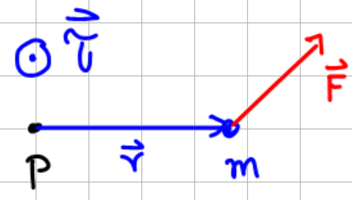
$$|\vec{\tau}| = |\vec{F}|\,|\vec{r}|\,|\sin\phi|$$

(ii) direction of $\vec{\tau}$: perpendicular to $\vec{r}$ - $\vec{F}$ plane

(iii) orientation of $\vec{\tau}$: according to right-hand (RH) rule

(i) – (iii) can be summarized by using the vector cross product (see math addendum, Sec. 1.8):

$$\underset{\text{middle finger}}{\vec{\tau}} = \underset{\text{thumb}}{\vec{r}} \times \underset{\text{index finger}}{\vec{F}}$$

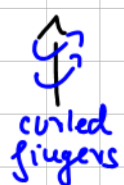


The rotational equation of motion can be generalized accordingly:

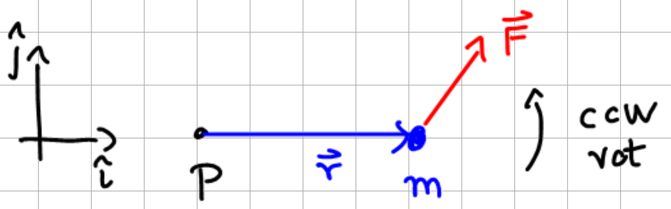
$$\boxed{I \vec{\alpha} = \vec{\tau}}$$

with $\vec{\alpha} = \frac{d}{dt} \vec{\omega}$

and $\vec{\omega}$ being a vector with
magnitude = | angular velocity |
direction/orientation ≙ rotation axis
simplified RH rule



e.g. counterclockwise (ccw) rotation of ball in x-y plane



$\vec{\omega} = (0, 0, \omega_z)$

with $\omega_z = \dot{\theta} > 0$

$\vec{\alpha} = (0, 0, \alpha_z) = (0, 0, \dot{\omega}_z)$

$\vec{\tau} = (0, 0, \tau_z)$

$\tau_z = I \alpha_z$

$(I = m r^2)$

a cw rotation about z-axis corresponds to

$\omega_z < 0$