Work and energy

One motivation to introduce work + energy: learn about the motion of an object without solving Newton's law of motion.

Work

⇨ 1D motion with constant force

recap: $a = \frac{F}{m}$

$$v(t) = v_0 + at$$

$$x(t) = x_0 + v_0 t + \frac{a}{2} t^2$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

rewrite the latter equation:

$$v_0 = v(t_i) = v_i \quad i = \text{initial}$$

$$v(t_f) \equiv v_f \quad f = \text{final}$$

$$x - x_0 = x(t_f) - x(t_i) =: \Delta x$$

$$\Rightarrow \quad v_f^2 = v_i^2 + 2a\Delta x$$

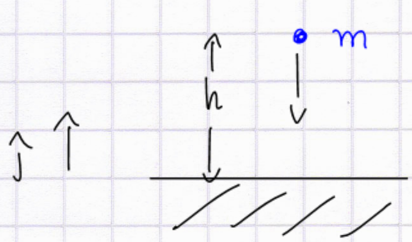
$$= v_i^2 + \frac{2F\Delta x}{m}$$

now define work (for this case). $W := F\Delta x$

units: $[W] = Nm = J$ (oule)

write $v_f^2 = v_i^2 + \frac{2W}{m}$

example (i): free fall with $v_i = 0$ (no air drag)



$$F_y = -mg$$

$$\Delta y = y(t_f) - y(t_i) = 0 - h = -h$$

$\hookrightarrow$ $W = F_y \Delta y = (-mg)(-h) = + mgh$

$$v_f^2 = \frac{2W}{m} = 2gh$$

(i.e. calculate final speed w/o going through full solution of Newton-2)

example (ii) : a variant of ex. (i) : $v_i > 0$ (i.e. throw rock vertically upward, consider displacement for reaching the max. height)

$$F_y = -mg$$

$$\Delta y = y(t_f) - y(t_i) = y_{max} - h$$

$\hookrightarrow$ $W = F_y \Delta y = -mg(y_{max} - h) < 0$

$$v_f^2 = v_i^2 - 2g(y_{max} - h)$$

$$\stackrel{!}{=} 0 \quad \Leftrightarrow \quad v_i^2 = 2g(y_{max} - h)$$

condition for max. height

$$\Leftrightarrow \quad y_{max} = h + \frac{v_i^2}{2g}$$

Discussion

(i) If F acts in direction of displacement, $W > 0$.

(ii) If F acts against displacement, $W < 0$.

(iii) Positive work speeds up object.

(iv) Negative work slows down object.

(v) Work is done by force.

Kinetic energy

defined by: $KE = \frac{m}{2} v^2$

use it to rewrite $v_f^2 = v_i^2 + \frac{2W}{m}$

work-energy theorem

into $\boxed{W = \frac{m}{2} v_f^2 - \frac{m}{2} v_i^2 = KE_f - KE_i = \Delta KE}$

note on units: $[KE] = J$

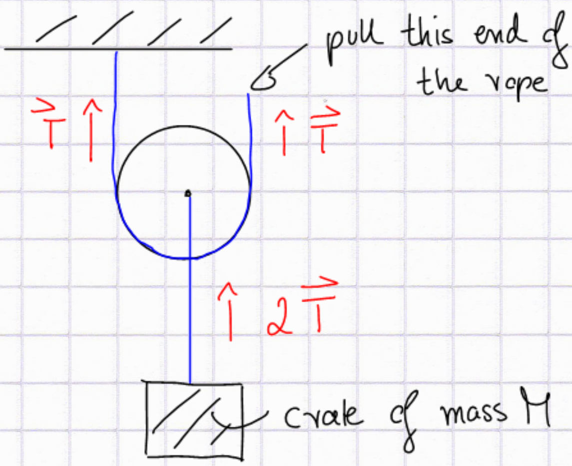
symbolic way of denoting work-energy theorem:

$$\boxed{KE_i} \xrightarrow{W} \boxed{KE_f}$$

t_i $\qquad$ t_f

note the conceptual difference between work and kinetic energy: while the latter (as well as potential energy to be discussed later) is a property of the object under study, the former isn't: work is what an agent (a force) is doing to the object

Another example: block and tackle device (book, Fig 6.8)



$F_{applied} = T$

in (static) equilibrium $F_{applied}$ balances weight of crate

$$Mg = 2T$$

- person lifts his end of the rope through Δx

  $\Rightarrow$ pulley (and crate) lifted by $\frac{\Delta x}{2}$

$$\Rightarrow \quad W_{on\ crate} = 2T \times \frac{\Delta x}{2} = T\Delta x = F_{applied}\,\Delta x = W_{person\ on\ rope}$$

the device amplifies (doubles) the force acting on the crate, but one still needs to do the same work (i.e. one gets the same work done from a smaller force if it acts over a proportionally longer distance).

now assume that the crate is lifted uniformly: $\Delta KE = 0$

(since $v_i = v_f$)

$\hookrightarrow$ work-energy theorem: $\Delta KE = W = 0$

Q: is there an inconsistency?

A: no, we must not forget about gravity:

$$F^{net}_{y_{\text{on crate}}} = 2T - Mg = 0 \iff W_{net} = W_{lift} + W_{grav} = 0$$
$$= \Delta KE \checkmark$$

i.e. work-energy theorem applies for total (net) work.

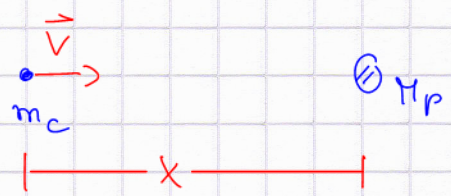
We have to address two complications:

(i) non-constant forces

(ii) forces which are not parallel to displacement

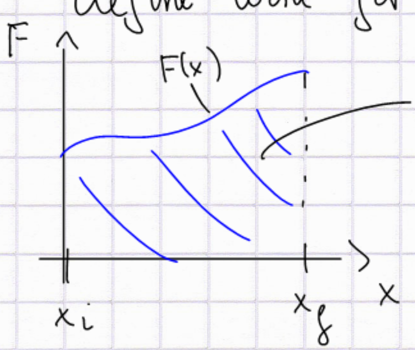
Complication (i) (discuss it only in 1D world)

example: a comet heading directly toward a planet (hopefully not toward Earth)



$$|F_{\text{on comet}}| = \frac{G m_c M_p}{x^2} =: F(x)$$

define work for non-constant forces as area under $F(x)$ curve

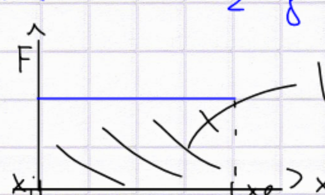


$$W = \int_{x_i}^{x_f} F(x)\,dx$$

one can show that work-energy theorem still holds, i.e.

$$W = \int_{x_i}^{x_f} F(x)\,dx = \frac{m}{2} v_f^2 - \frac{m}{2} v_i^2 = \Delta KE$$

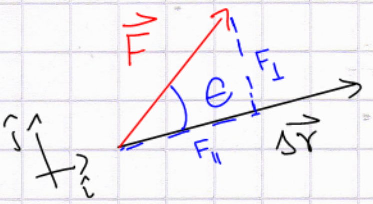
note that previous definition $W = F\Delta x$ for constant F is consistent with this more general one:



$W = F\Delta x$ (area of a rectangle)

⑤

- Complication (ii) (discuss it only for constant forces)



decompose $\vec{F} = F_\parallel \hat{\imath} + F_\perp \hat{\jmath}$

$= F\cos\theta\,\hat{\imath} + F\sin\theta\,\hat{\jmath}$

define work: $W = F_\parallel \Delta r = F \Delta r \cos\theta$

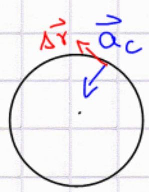
special cases: $W = +F\Delta r$ for $\theta = 0$

$W = -F\Delta r$ for $\theta = \pi$

$W = 0$ for $\theta = \frac{\pi}{2}$

i.e. no work is done if $\vec{F} \perp \Delta\vec{r}$.

example: uniform circular motion



$\vec{F}_{net} = m\vec{a}_c$ (toward centre)

$\Delta\vec{r} = \vec{v}\,\Delta t$ (for small Δt)

we showed (in tutorial) that $\vec{v} \perp \vec{a}_c$

$\Rightarrow \vec{F}_{net} \perp \Delta\vec{r}$; $W = 0$

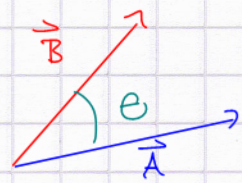
(note that this is consistent with work-energy theorem: $\Delta KE = 0$ since $v = \text{const}$)

math aside: define dot product of two vectors

$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$

$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$

$\vec{A}\cdot\vec{B} := |\vec{A}|\,|\vec{B}|\cos\theta$



express $\vec{A}\cdot\vec{B}$ (in 2D) in terms of components:

$\vec{A}\cdot\vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath})\cdot(B_x\hat{\imath} + B_y\hat{\jmath})$

$= A_xB_x\underbrace{\hat{\imath}\cdot\hat{\imath}}_{=\cos 0 = 1} + A_xB_y\underbrace{\hat{\imath}\cdot\hat{\jmath}}_{=\cos\frac{\pi}{2} = 0} + A_yB_x\underbrace{\hat{\jmath}\cdot\hat{\imath}}_{=} + A_yB_y\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{=\cos 0 = 1}$

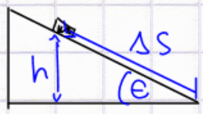
$$= A_x B_x + A_y B_y \quad (\text{add } A_z B_z \text{ in 3D})$$

summary :
- In 2D and 3D world work (for a constant force $\vec{F}$) is defined as $W = \vec{F} \cdot \Delta\vec{r}$
- One can show that work-energy theorem still holds:

$$W = \vec{F} \cdot \Delta\vec{r} = \frac{m}{2} v_f^2 - \frac{m}{2} v_i^2 = \Delta KE$$

Potential energy

- recap: work in free fall (see pp 1): $W = mgh$
- now consider sliding down on an inclined plane (w/o friction)



$\sin\theta = \frac{h}{\Delta s}$

$$W = F_{net}\, \Delta s = mg \sin\theta\, \Delta s$$
$$= mg \frac{h}{\Delta s} \Delta s = mgh$$

(same result!)

variations

ex ①: free fall from y_i to y_f

$$W = F_y\, \Delta y = (-mg)(y_f - y_i) = mg y_i - mg y_f$$

ex ②: sliding down on inclined plane from initial height y_i to final height y_f:

$$W = F_{net}\, \Delta s = mg \sin\theta \frac{y_i - y_f}{\sin\theta} = mg y_i - mg y_f$$

(still the same result!)

now define potential energy (PE) (for these examples)

$$PE = mgy$$

$\hookrightarrow$ $W = PE_i - PE_f$

turn it around: $\Delta PE = PE_f - PE_i = -W$

and combine it with work-energy theorem:

$$W = \Delta KE = -\Delta PE$$

$$\Leftrightarrow \quad \Delta KE + \Delta PE \equiv \Delta TE = 0$$

$$\Leftrightarrow \quad \boxed{KE_i + PE_i = KE_f + PE_f = TE = \text{const}}$$

total (mechanical) energy TE is conserved

reconsider free fall

$@t_i: \quad \left.\begin{matrix} KE_i = 0 \\ PE_i = mgy_i \end{matrix}\right\} \quad TE = mgy_i$

$\qquad\qquad\qquad\qquad\qquad\qquad \|$

$@t_f: \quad \left.\begin{matrix} KE_f = \frac{m}{2}v_f^2 \\ PE_f = mgy_f \end{matrix}\right\} \quad TE = mgy_f + \frac{m}{2}v_f^2$

$$\Rightarrow \quad \frac{m}{2}v_f^2 = mg(y_i - y_f) \qquad \text{if } y_i = h,\ y_f = 0$$

$$\Leftrightarrow \quad v_f = \sqrt{2g(y_i - y_f)} \overset{!}{=} \sqrt{2gh}$$

(obtain the same result for inclined plane)

Discussion

(i) Energy situation is the same for simple free fall and (frictionless) motion on inclined plane. What differs is the time it takes to get to the ground. To find those arrival times one has to look at complete solution of Newton-2 (i.e. one cannot get this information from energy consideration)

(ii) PE is interpreted as stored energy

release object $\Rightarrow$ convert PE to KE according to $\Delta KE = -\Delta PE$

(iii) For more general PEs go back to general definition of work (in 1D):

$$W = \int_{x_i}^{x_f} F(x)\,dx = \overbrace{PE_i - PE_f}^{\text{see above examples}}$$

change notation: $PE \to V(x)$

$$= V(x_i) - V(x_f) = -\left(V(x_f) - V(x_i)\right)$$

$$= V(x)\Big|_{x_i}^{x_f}$$

$\rightarrow$ identify potential energy as negative anti-derivative of force and use this as general definition

(note that like any anti-derivative $V(x)$ is only defined up to a constant)

$$\Longleftrightarrow \quad F(x) = -\frac{dV}{dx}$$

example (i): $V(y) = mgy$ (+ const)

$$\rightarrow F_y(y) = -\frac{dV}{dy} = -mg \quad \checkmark$$

i.e. previous ex. consistent with this more general definition

example (ii): $V(x) = \frac{k}{2}x^2$ (with constant k)

$$\rightarrow F_x(x) = -\frac{dV}{dx} = -kx \quad \text{(Hooke's law)}$$

We will see later that this force causes oscillatory motion.

example (iii): $V(r) = -\frac{GmM}{r}$ $\leftarrow$ general gravitational PE

$$\rightarrow F_r(r) = -\frac{dV}{dr} = GmM\frac{d}{dr}\left(\frac{1}{r}\right) = \underset{\uparrow \text{ attractive}}{-}\frac{GmM}{r^2}$$

(gravity!)

application: launching a spacecraft; i.e. shoot it vertically upward with an initial speed such that it doesn't come back.

$\rightarrow$ infer v_i from energy conservation theorem:

@ t_i: $KE_i = \frac{m}{2}v_i^2$ $\qquad$ $KE_f \approx 0$ ($v_f \approx 0$)

$PE_i = -\frac{GmM_E}{r_E}$ $\qquad$ $PE_f \approx 0$ ($r_f \to \infty$)

$$\rightarrow TE = KE_i + PE_i = KE_f + PE_f$$

$$\Longleftrightarrow \frac{m}{2}v_i^2 - \frac{GmM_E}{r_E} = 0$$

$$\rightarrow v_i = \sqrt{\frac{2GM_E}{r_E}} \approx 40{,}000 \text{ km/h}$$

("escape velocity")

(iv) Q: How general are the work-energy and the energy-conservation theorems?

A: $W = \Delta KE$ holds for all forces that accelerate the object under study according to Newton-2.

$KE + PE = TE = const$ is more restricted (less general).

one cannot define PE for all physical forces, e.g. frictional and drag forces cannot be written as derivatives of PE-functions

One calls forces that fulfill $F(x) = -\frac{dV}{dx}$

"conservative forces".

For conservative forces total mechanical energy is conserved ($\Delta TE = 0$).

Non-conservative forces (such as air drag and friction) convert mechanical energy into heat. One finds that for them work depends on the actual path the object travels – not just the initial and final positions as in the examples above.