# PHYS 3090: Homework 2 (due Friday Sept. 25)

**Problem 1 (20 points):** Consider a double-spring system, with equal masses $m$ and spring constants $k$ and $\frac{3}{2}k$, where $x_1$ and $x_2$ measure the displacements from the equilibrium (see Figure 1 left).

- What is the potential energy $V$? **(5 points)**
- Determine the normal frequencies and the normal modes. **(5 points)**
- Suppose at $t = 0$, the system had the initial condition

  $$x_1(0) = d\,, \quad x_2 = 0\,, \quad \dot{x}_1(0) = \dot{x}_2(0) = 0\,. \tag{1}$$

  Using eigenvalue methods, determine $\vec{x}(t)$ for $t > 0$. **(5 points)**
- Compute the linear momentum $p_1$ and $p_2$ of each of the two masses as a function of $t$. **(5 points)**

**Problem 2 (30 points):** Consider circular configuration of three masses shown in Figure 1 (right). All three masses (with mass $m$) and all three springs (with spring constant $k$) are fixed to move along the circle of radius $R$. Let the variables $(\theta_1, \theta_2, \theta_2)$ be the angular displacements of each mass from its equilibrium position.

- What is the potential energy $V$ in terms of $k, R$, and $\theta_i$? **(5 points)**
- Note that for circular motion, Newton's second law can be expressed as

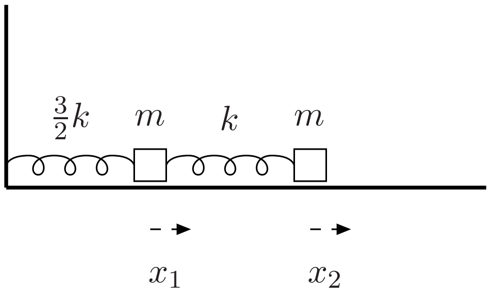
  $$m\ddot{\theta}_i = -\frac{1}{R^2}\frac{\partial V}{\partial \theta_i}\,.$$

  Write the equation of motion for this system as $\ddot{\vec{\theta}} = -U\vec{\theta}$, where $\vec{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$ and $U$ is a $3 \times 3$ matrix.

  What is $U$ in terms of $k$ and $m$? **(5 points)**
- What are the normal frequencies and normal modes for this system? **(5 points)**
- Suppose at time $t = 0$ all masses begin at their equilibrium positions and we give mass 1 a "kick" so that it has velocity $v$ (and the other masses begin at rest). This corresponds to an initial condition:

  $$\theta_1(0) = \theta_2(0) = \theta_3(0) = 0, \qquad R\,\dot{\theta}_1(0) = v, \quad \dot{\theta}_2(0) = \dot{\theta}_3(0) = 0\,. \tag{2}$$

  Using eigenvalue methods, determine $\vec{\theta}(t)$ for $t > 0$. **(10 points)**
- Compute the angular momentum $L_1$, $L_2$, and $L_3$ for each of the three masses as a function of $t$. Show that the total angular momentum $L_1 + L_2 + L_3$ is constant in time. **(5 points)**



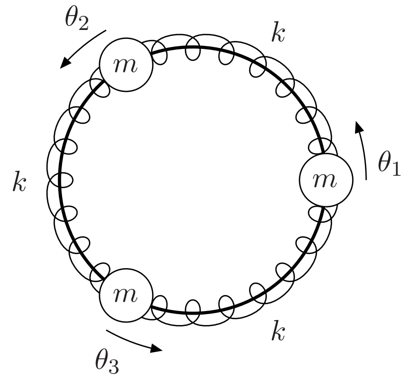




Figure 1: Left: Double spring system with spring constants $k$ and $\frac{3}{2}k$ and masses $m$ on a frictionless surface. Right: three masses $m$ connected by springs in a ring configuration.

**Problem 3 (25 points):** In the quantum mechanical description of angular momentum, the **Pauli matrices**

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

play an important role. Defining $\vec{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$, any real three-component vector $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ can be expressed as a $2 \times 2$ traceless Hermitian matrix $M$:

$$M = \vec{v} \cdot \vec{\boldsymbol{\sigma}} = \sum_{i=1}^{3} v_i \sigma_i = \begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix} \tag{4}$$

- Show that $(\vec{v} \cdot \vec{\boldsymbol{\sigma}})^2 = |\vec{v}|^2 \, \mathbb{1}$ for any $\vec{v}$, where $\mathbb{1}$ is the $2 \times 2$ identity matrix. **(5 points)**
- Show that $\exp(iM) = \cos|\vec{v}| \, \mathbb{1} + i(\hat{v} \cdot \vec{\boldsymbol{\sigma}}) \sin|\vec{v}|$. (Note: $\hat{v} = \vec{v}/|\vec{v}|$.) Use your result to compute $\exp\begin{pmatrix} i\sqrt{2}\pi & -\sqrt{2}\pi \\ \sqrt{2}\pi & -i\sqrt{2}\pi \end{pmatrix}$. **(10 points)**
- Consider a new, more general matrix $M = v_0 \mathbb{1} + \vec{v} \cdot \vec{\boldsymbol{\sigma}} = \begin{pmatrix} v_0 + v_z & v_x - iv_y \\ v_x + iv_y & v_0 - v_z \end{pmatrix}$, where now $M$ is no longer traceless. What is $\exp(iM)$? **(10 points)**