PHYS 3090: Homework 4 (due Friday Oct. 16)

Problem 1: Identify the singular points in the following functions and demonstrate whether they are poles, essential singularities, or removable singularities.

- $f(z) = \cos(z + 1/z)$
- $f(z) = \frac{z^3+6z^2+5z-12}{3z^2-6z+3}$
- $f(z) = \frac{z^2-4}{z+2}$
- $f(z) = \cot(z) - 1$

If any of the above functions have poles, determine their order.

Problem 2: Compute the integral $\int_{z_1}^{z_2} dz\, z^2$, where $z_1 = 0$ and $z_2 = 1 + 2i$, along the following contours:

- First vertically from $z = 0$ to $z = 2i$, and then horizontally from $z = 2i$ to $z = 1 + 2i$.
- Along the straight line defined by $y = 2x$.
- Along a parabolic curve defined by $y = 2x^2$.

Verify that you obtain the same result for each path. (*Hint:* For an arbitrary path defined by $y = y(x)$, you can write $dz = dx + idy = dx + i\frac{dy}{dx}dx$.)

Problem 3: Consider the function $f(z) = g(z)/h(z)$, where $g(z)$ is analytic for all z. Suppose $f(z)$ has a simple pole at $z = a$.

- Show that the residue at $z = a$ is given by $\text{Res}\, f(a) = g(a)/h'(a)$. (*Hint:* Taylor expand something!)
- For the function $f(z) = \frac{e^z-1}{e^z+1}$, determine the locations and orders for all poles and compute their residues.