PHYS 3090: Homework 6 (due Monday Nov. 2)

Problem 1: Consider an analytic function $f(z) = u(x,y) + iv(x,y)$. Show that if u, v are expressed in polar coordinates, then the Cauchy-Riemann relations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \qquad \textbf{(10 points)} \tag{1}$$

Problem 2: Compute $\int_0^{2\pi} d\theta \, \frac{1}{(2-\cos\theta)^2}$ using contour integration. **(10 points)**

Problem 3: Compute the principal value of $\int_{-\infty}^{\infty} dx \, \frac{\cos x - 1}{x^2}$ by contour integration. **(10 points)**

Problem 4: Compute $\int_0^{\infty} dx \, \frac{1}{1+x^{100}}$ by contour integration. **(10 points)**

Hint 1: You may use the result from problem 3 in HW 4 to compute the residues.

Hint 2: Your intermediate steps may involve a finite geometric series, which can be summed as follows

$$\sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1-x}. \tag{2}$$

Problem 5: Compute the integral $I = \int_{-\infty}^{\infty} dx \, \frac{e^x}{e^{3x}+1}$. We are going to evaluate I using a rectangular contour C shown in Fig. 1, with sides labeled by paths P_1 - P_4. **(15 points)**

- Show that $f(z) = \frac{e^z}{e^{3z}+1}$ has poles along the imaginary axis at $z = \frac{\pi i}{3} + \frac{2\pi i n}{3}$, where $n = 0, \pm 1,$ *etc.*
- The original integral is $I = \lim_{R\to\infty} \int_{P_1} dz \, f(z)$. Show that $\lim_{R\to\infty} \int_{P_3} dz \, f(z) = -e^{2\pi i/3} I$.

  Hint: Use $z = x + \frac{2\pi i}{3}$ along P_3.
- Show that $\lim_{R\to\infty} \int_{P_2} dz \, f(z) = \lim_{R\to\infty} \int_{P_4} dz \, f(z) = 0$.
- Combining these results, you have

$$\lim_{R\to\infty} \int_{P_1} dz\, f(z) + \int_{P_2} dz\, f(z) + \int_{P_3} dz\, f(z) + \int_{P_4} dz\, f(z) = \left(1 - e^{2\pi i/3}\right) I = \oint_C dz\, f(z). \tag{3}$$

  Now use the residue theorem to evaluate the right-hand side and solve for the original integral I.

Note: This problem is similar to Example 2.22 in the textbook.

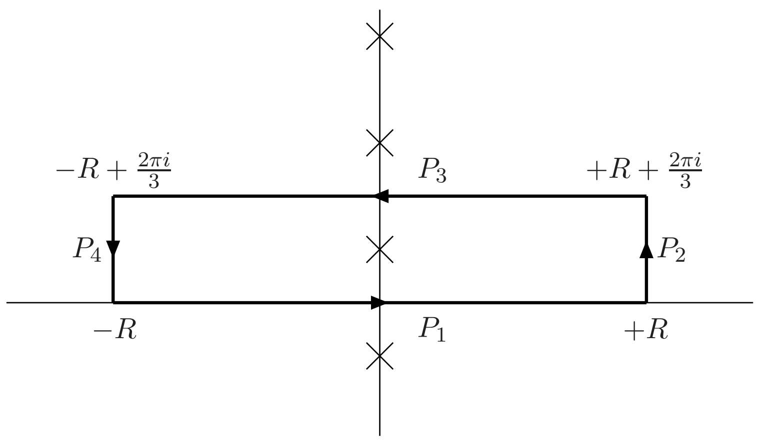

Figure 1: Rectangular contour C defined by the corners $z = \pm R$ and $z = \pm R + \frac{2\pi i}{3}$, and sides P_1-P_4. The crosses show the poles.