PHYS 3090: Homework 9 (due Monday Dec. 7)

Problem 1: The wave function for a quantum mechanical particle of mass m satisfies the Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\hbar\frac{\partial\Psi}{\partial t}\,. \tag{1}$$

Consider an initial condition $\Psi(x,0) = N\,e^{-\alpha^2x^2/2}$, where $N = \sqrt{\alpha/\sqrt{\pi}}$.

- Compute $\Psi(x,t)$. *Hint:* First, Fourier transform Eq. (1) with respect to x and solve for $\tilde{\Psi}(k,t)$.
- Compute $|\Psi(x,t)|^2$. Sketch what the solution looks like at time $t = 0$ and at a later time t_1, as a function of x.
- Determine the uncertainties Δx and Δp. What is happening to the wave packet?
- What is the dispersion relation for Eq. (1)? What is the wave velocity as a function of k?

This problem illustrates that systems with a nonlinear dispersion relation (i.e., $\omega/k \neq$ constant) exhibit wave packet spreading since different Fourier modes are moving with a different wave velocity.

Problem 2: Consider the wave equation for an infinite string

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}\,, \tag{2}$$

with the initial condition $y(x,0) = f(x)$, where $f(x)$ is an arbitrary function, and $\dot{y}(x,0) = 0$. *Using Fourier transform*, show that the solution for $t > 0$ is given by

$$y(x,t) = \tfrac{1}{2}\big(f(x-vt) + f(x+vt)\big)\,. \tag{3}$$

Hint: First, compute the Fourier transform $Y(k,t)$ in terms of $F(k)$, the Fourier transform for $f(x)$.

This shows that an initial displacement in the string separates into two oppositely-moving traveling waves that maintain the initial shape $f(x)$, each with half the amplitude.