PHYS 3090: Homework 1 (due Wednesday Sept. 20)

The following problems cover some basics of complex variables.

Problem 1: Let $z = -1 + i$. Compute the following:

(a) $\arg z$ **(1 points)**

(b) $|z|$ **(1 points)**

(c) $\ln(z)$, taking the principal branch with $0 \leq \arg(z) < 2\pi$ **(2 points)**

Problem 2: Express $z = \frac{4-2i}{1+i}$ in the form $z = x + iy$. **(2 points)**

Problem 3: Show that for any complex numbers z_1 and z_2, the following are true:

(a) $|z_1 z_2| = |z_1|\,|z_2|$ **(2 points)**

(b) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ **(2 points)**

(c) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$ **(2 points)**

Problem 4: Find the unique solutions of the equation $z^4 = -16i$. **(3 points)**

Any complex function can be expressed as $f(z) = u(x, y) + i\, v(x, y)$, where u, v are purely real functions and $z = x + iy$. The following problems practice finding u and v.

Problem 5: Find $u(x, y)$ and $v(x, y)$ for the complex function $f(z) = e^{1/z}$. **(3 points)**

Problem 6: For the complex function $f(z) = \tan(z)$, show the following: **(3 points)**

$$u(x, y) = \frac{\sin(2x)}{\cos(2x) + \cosh(2y)}\,, \quad v(x, y) = \frac{\sinh(2y)}{\cos(2x) + \cosh(2y)} \tag{1}$$

This problem practices computing complex integrals through direct integration. You should get the same answer for (a)-(c), which follows as a consequence of Cauchy's theorem since $f(z) = z^2$ is an entire function.

Problem 7: Compute the integral $\int_{z_1}^{z_2} dz\, z^2$, where $z_1 = 1$ and $z_2 = i$, along each of the following contours:

(a) First horizontally from $z = 1$ to $z = 0$, and then vertically from $z = 0$ to $z = i$. **(3 points)**

(b) Along the straight line defined by $y = 1 - x$. Hint: For an arbitrary path defined by $y = y(x)$, you can write $dz = dx + idy = dx + i\frac{dy}{dx}dx$. **(3 points)**

(c) Along the unit circle defined by $|z| = 1$. Hint: write z in polar form and use $dz = \frac{dz}{d\theta}d\theta$. **(3 points)**