Section 3: slack and surplus variables, LP in standard form, free variables, matrix form of LP problem in standard form

A. LP Problem in Standard Form

Def. 3.1: An LP is said to be in **STANARD FORM** iff it is of the form

minimize $z=c_1x_1+c_2x_2+\ldots+c_nx_n$

subject to

$$
\begin{array}{ccccccccl}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \quad (1) \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \quad (2) \\
\vdots & & \vdots & & & & & & \vdots \quad\ \ \vdots \\
\vdots & & \vdots & & & & & & \vdots \quad\ \ \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \quad (m)
\end{array}
$$

$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$

where

(1) $b_1, b_2, \ldots, b_m$ are nonnegative constants

(2) a_{ij} $i=1,\ldots,m;j=1,\ldots,n$ are constants

(3) $c_1,\ldots,c_n$ are constants

(4) $x_1,\ldots,x_n$ are variables

1. properties of standard LP are:

a. constraints are equations with right hand side nonnegative

b. variables are all nonnegative

c. objective function is minimized

We will show that any LP problem can be put into standard form. We could prove this as a theorem but we will not. Instead we shall look at examples where we introduce the techniques for putting an LP into standard form.

The Reddy Mikks Co. example

LP problem $\max z = 3x_E + 2x_I$

s.t.

$x_E + 2x_I \leq 6$ (1)

$2x_E + x_I \leq 8$ (2)

$-x_E + x_I \leq 1$ (3)

$x_I \leq 2$ (4)

$x_E \geq 0,\ x_I \geq 0$

We wish to turn it into standard form.

Step 1: Change max to min by

$\min z = -(3x_E + 2x_I)$

$\min z = -3x_E - 2x_I$

If we minimize this objective function than we maximize the original objective function.

e.g. if $h(x)$ is a function and the largest value occurs at $x=10$ and $h(10)=50$ then the smallest value of $-h(x)$ is -50 and occurs at $x=10$. Conversely if

$$-50=h(10)\leq -h(x) \text{ for all } x \text{ then } 50=h(10)\geq h(x) \text{ for all } x$$

Step 2: **Introduce SLACK VARIABLES to turn inequalities into equalities**

Def. 3.2: **Given a ≤ inequality whose right hand side is positive, a SLACK VARIABLE is a new nonnegative variable y which we introduce into the model to convert the ≤ inequality into an equality by adding y from the left hand side ("pull in the slack of the inequality and put it into the slack variable")**

In the Reddy Mikks problem, we will hav to introduce 4 different slack variables y_1, y_2, y_3, y_4, one for each inequality.

Standard form of Reddy Mikks LP:

$$\min z = -3x_E - 2x_I$$

s.t.

$$
\begin{array}{llr}
x_E + 2x_I + y_1 & = 6 & (1) \\
2x_E + x_I \quad + y_2 & = 8 & (2) \\
-x_E + x_I \quad\quad + y_3 & = 1 & (3) \\
x_I \quad\quad\quad\quad + y_4 & = 2 & (4)
\end{array}
$$

$$x_E \geq 0,\ x_I \geq 0,\ y_i \geq 0\ \ i=1,\dots,4$$

Def. 3.3: Given a ≤ inequality whose right hand side is positive, a **SURPLUS VARIABLE** is a new nonnegative variable y which we introduce into the model to convert the ≥ inequality into an equality by subtracting y from the left hand side ("pull in the surplus of the inequality and put it into the surplus variable")

Recall: production to meet demand at minimum cost

$$\min z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

s.t.

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq \delta_1 \qquad (1)$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq \delta_2 \qquad (2)$$

...

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq \delta_m \qquad (m)$$

$x_i \geq 0,\ i=1\ldots n$

in standard form becomes

$$\min z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

s.t.

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n - y_1 = \delta_1 \qquad (1)$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \qquad - y_2 = \delta_2 \qquad (2)$$

$$\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \quad \ldots \quad - y_m = \delta_m \qquad (m)$$

$$x_i \geq 0,\ i=1\ldots n;\ y_j \geq 0,\ j=1,\ldots,m$$

Bus scheduling problem:

Minimize $z = x_1 + \ldots + x_6$

s.t.

$x_1 + x_6 \geq 4$ (1)

$x_1 + x_2 \geq 8$ (2)

$x_2 + x_3 \geq 10$ (3)

$x_3 + x_4 \geq 7$ (4)

$x_4 + x_5 \geq 12$ (5)

$x_5 + x_6 \geq 4$ (6)

$x_i \geq 0, i=1,\ldots,6$

in standard form becomes

Minimize $z = x_1 + \ldots + x_6$

s.t.

$x_1 + x_6 - y_1 = 4$ (1)

$x_1 + x_2 - y_2 = 8$ (2)

$$x_2 + x_3 \qquad - y_3 \qquad = 10 \qquad (3)$$

$$x_3 + x_4 \qquad - y_4 \qquad = 7 \qquad (4)$$

$$x_4 + x_5 \qquad - y_5 \qquad = 12 \qquad (5)$$

$$x_5 + x_6 \qquad - y_6 = 4 \qquad (6)$$

$$x_i \geq 0,\ i=1,\dots,6;\ y_i \geq 0,\ i=1,2,\dots,6$$

Note: The # of surplus variables= # constraints with ≥ with the RHS ≥ 0. If the RHS <0, then multiply through by –1 (which reverses the inequality to a ≤ with the RHS > 0) so that we use a slack variable for these constraints.

Example 3.1 (of slack and surplus variables)

$\min z = 5x_1 - 3x_2 + 2x_3$

s.t.

$$
\begin{aligned}
x_1 - x_2 &\le 10 \quad (1)\\
x_1 + x_3 &\ge 7 \quad (2)\\
x_1 + x_2 + x_3 &= 19 \quad (3)\\
x_1 &\ge 2 \quad (4)\\
x_2 + x_3 &\ge 1 \quad (5)\\
x_i \ge 0,\ i &= 1,2,3
\end{aligned}
$$

In standard form:

$\min z = 5x_1 - 3x_2 + 2x_3$

s.t.

$$x_1 - x_2 + y_1 = 10 \quad (1) \text{ slack}$$

$$x_1 + x_3 - y_2 = 7 \quad (2) \text{ surplus}$$

$$x_1 + x_2 + x_3 = 19 \quad (3) \text{ none}$$

$$x_1 - y_3 = 2 \quad (4) \text{ surplus}$$

$$x_2 + x_3 - y_4 = 1 \quad (5) \text{ surplus}$$

$$x_i \geq 0,\ i=1,2,3;\ y_j \geq 0,\ j=1,\ldots,4$$

Example 3.2: (RHS not all nonnegative)

$$\min z = x_1 - x_2 + x_3$$

s.t.

$$x_1 - x_3 = -1 \quad (1)$$

$$x_1 - x_2 \geq -1 \quad (2)$$

$$x_1 + x_2 + x_3 \leq 14 \quad (3)$$

$$-x_1 \qquad\qquad\qquad \le -5 \qquad (4)$$

$$x_i \ge 0,\ i=1,2,3$$

Step 1: Rewrite constraints so that RHS is nonnegative. Note that inequalities change direction when multiplied by -1.

$$-x_1 \qquad + x_3 \qquad = 1 \qquad (1)\ (1)\cdot -1$$

$$-x_1 + x_2 \qquad \le 1 \qquad (2)\ (2)\cdot -1$$

$$x_1 + x_2 + x_3 \le 14 \qquad (3) \text{ ok as is}$$

$$x_1 \qquad\qquad \ge 5 \qquad (4)\ (4)\cdot -1$$

Step 2: Put in slack or surplus variable as required.

$$\min z = x_1 - x_2 + x_3$$

s.t.

$$-x_1 \qquad + x_3 \qquad = 1 \qquad (1) \text{ none}$$

$-x_1 + x_2 \qquad +y_1 \qquad = 1$ (2) slack

$x_1 + x_2 \quad +x_3 \quad +y_2 \qquad = 14$ (3) slack

$x_1 \qquad -y_3 \quad = 5$ (4) surplus

$x_i \geq 0,\ i=1,2,3;\ y_i \geq 0,\ i=1,2,3$

B. Free Variable

Def. 3.4: A variable is called a **FREE VARIABLE** if it is unconstrained in sign.

Example 3.3: free variable

$\min z = x_1 + 2x_2 - 1/2\ x_3$

s.t.

$x_1 - x_2 \quad + x_3 \quad = 17$ (1)

$x_1 \qquad \leq 25$ (2)

$$x_2 \leq 19 \quad (3)$$

$$x_1 + x_3 \geq 5 \quad (4)$$

$x_i \geq 0$, $i=1,2$; x_3 unconstrained in sign

How do we deal with free variables in order to convert the LP into standard form?

<u>Method 1:</u> Let $x_3=u_1-v_1$ where $u_1\geq 0$ and $v_1\geq 0$.

<u>Note:</u> u_1 and v_1 are not unique; e.g. if $x_3=5$, we take $u_1=12$ and $v_1=7$ or $u_1=17$ and $v_1=12$. If $x_3=u_1-v_1$ then replacing u_1 and v_1 by u_1+k and v_1+k will also work for any k such that $u_1+k\geq 0$ and $v_1+k\geq 0$. That at least one such pair exists is easy:

if $x_3\geq 0$ take $u_1=x_3$ and $v_1=0$ since $u_1=x_3\geq 0$ and $v_1=0\geq 0$

if $x_3< 0$ take $u_1=0$ and $v_1=-x_3$ since $u_1-v_1=0-(-x_3)=0+x_3=x_3$ and

$u_1 = 0 \geq 0$ and since $x_3 < 0$, then $-x_3 > 0$ so $v_1 = -x_3 \geq 0$.

In standard form:

$$\min z = x_1 + 2x_2 - 1/2\, u_1 + 1/2\, v_1$$

s.t.

$$x_1 - x_2 + u_1 - v_1 = 17 \quad (1)$$

$$x_1 + y_1 = 25 \quad (2)$$

$$x_2 + y_2 = 19 \quad (3)$$

$$x_1 + u_1 - v_1 - y_3 = 5 \quad (4)$$

$$x_i \geq 0,\ i=1,2;\ u_1 \geq 0;\ v_1 \geq 0;\ y_i \geq 0,\ i=1,2,3$$

Method 2: For the moment, ignore the free variable issue and write the LP in standard form

$\min z = x_1 + 2x_2 - 1/2\ x_3$

s.t.

x_1	$-x_2$	$+x_3$				$= 17$	(1)
x_1	$+$		y_1			$= 25$	(2)
	x_2			$+y_2$		$= 19$	(3)
x_1		$+x_3$			$-y_3$	$= 5$	(4)

$x_i \geq 0,\ i=1,2;\ y_i \geq 0,\ i=1,2,3;\ x_3$ unrestricted in sign

Solve one of the equations for x_3 in terms of the other variables (which are all nonnegative). In this case, say equation (4)

$$x_3 = 5 - x_1 + y_3$$

and substitute wherever x_3 appears in the other constraints.

Note that equation (4) disappears as a constraint since we are using method of substitution.

So in our example we get

$$\min z = x_1 + 2x_2 - 1/2\,(5 - x_1 + y_3)$$

s.t.

$$
\begin{array}{llll}
x_1 - x_2 + (5 - x_1 + y_3) & = 17 & (1) \\
x_1 + y_1 & = 25 & (2) \\
x_2 + y_2 & = 19 & (3) \\
x_1 + x_3 + (5 - x_1 + y_3) - y_3 & = 5 & (4)
\end{array}
$$

$$x_i \geq 0,\ i=1,2;\ y_i \geq 0,\ i=1,2,3$$

which simplifies to

$$\min z = 3/2x_1 + 2x_2 - 1/2\, y_3 - 5/2$$

s.t.

$$
\begin{array}{llllll}
 & -x_2 & & +y_3 & = 12 & (1) \\
x_1 & + & y_1 & & = 25 & (2) \\
 & x_2 & & +y_2 & = 19 & (3)
\end{array}
$$

$x_i \geq 0,\ i=1,2;\ y_i \geq 0,\ i=1,2,3$

Note that the objective function is not linear (a problem) because of the –5/2. However, this function reaches its optimal value at the same point that the objective function

$z=3/2x_1 +2x_2-1/2\ y_3$ reaches its optimal value so we replace the nonlinear objective function by the linear objective function to get the problem in standard form.

$\min z = 3/2x_1 + 2x_2 - 1/2\, y_3$

s.t.

$$-x_2 + y_3 = 12 \quad (1)$$

$$x_1 + y_1 = 25 \quad (2)$$

$$x_2 + y_2 = 19 \quad (3)$$

$$x_i \geq 0,\ i=1,2;\ y_i \geq 0,\ i=1,2,3$$

What are the advantages and disadvantages of the two methods?

(1) Method (1) is simpler to apply (simple replacement) while Method (2) involves solving equations (set of equations if there is more than one free variable) and substitution.

(2) Method (2) has less variables and fewer constraints (a big advantage which sometimes makes up for the extra work involved in solving for the free varable.

(3) Method (1) can more easily be programmed; but method (2) will solve faster once implemented.

C. Matrix Form of Standard LP

1. $A=(a_{ij})$ i=1,...,m; j=1,...,n is an mxn matrix of consants

2. $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ **is a nx1 matrix (i.e. a column vector of length n) of variables.**

3. $\underline{c} = (c_1, c_2, \ldots, c_n)$ **is a 1xn matrix (i.e. a row vector of length n) of constants**

4. $\underline{\mathbf{b}} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ is an mx1 matrix (a column vector of length m) of nonnegative constants; i.e. $\underline{\mathbf{b}} \geq \underline{\mathbf{0}}$.

Then the LP problem in standard form in matrix notation is

$$\min \ \mathbf{z} = \underline{\mathbf{c}} \cdot \underline{\mathbf{x}}$$

s.t.

$$\mathbf{A}\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

$$\underline{\mathbf{x}} \geq \underline{\mathbf{0}}$$

with the above setup.

END OF SECTION 3