Uniform circular motion

general characteristics :

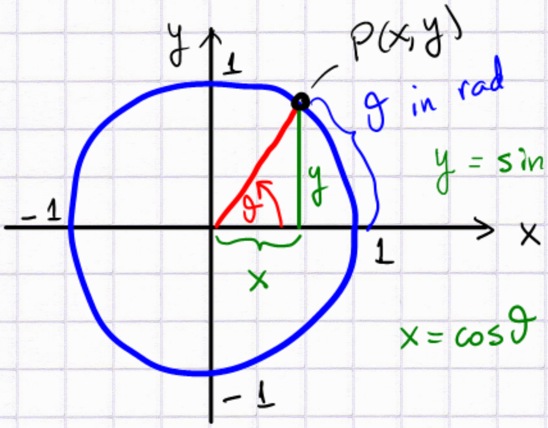
radius : $R = \text{const}$

period : T

frequency : $f = \frac{1}{T}$ (or $\nu = \frac{1}{T}$)

circular frequency : $\omega = 2\pi f = \frac{2\pi}{T}$

Use the unit circle



one revolution : $0 \leq \vartheta < 2\pi$

$$\vec{r}(t) = \underbrace{(R\cos\omega t)}_{x(t)}\,\hat{\imath} + \underbrace{(R\sin\omega t)}_{y(t)}\,\hat{\jmath}$$

why is this true ?

as $t \to 0 \,..\, T$

(one period = one revolution)

$\omega t = \frac{2\pi}{T} t$ goes

$0 \,..\, 2\pi$ ← once around on circle

Kinematics of circular motion:

- need to know how to differentiate $\begin{matrix}\sin\\ \cos\end{matrix}(\omega t)$ with respect to t

- chain rule : $\frac{d}{dx}\left[g(f(x))\right] = \left(\frac{dg}{dx}\right)\left(\frac{df}{dx}\right)$

Thus : $\frac{d}{dt}\left[\sin(\omega t)\right]$: $g = \sin$ $\quad f = \omega t$

$= \cos(\omega t) \cdot \omega$ $\quad g' = \sin$ $\quad f' = \omega$!

Likewise : $\frac{d}{dt}\left[\cos(\omega t)\right] = -\sin(\omega t) \cdot \omega$

Now we find from (1) $\vec{r}(t) = R\cos\omega t\ \hat{\imath} + R\sin\omega t\ \hat{\jmath}$

(2) $\vec{v}(t) \equiv \frac{d}{dt}\vec{r}(t) = -R\omega\sin\omega t\ \hat{\imath} + R\omega\cos\omega t\ \hat{\jmath}$

(3) $\vec{a}(t) \equiv \frac{d}{dt}\vec{v}(t) = -R\omega^2\cos\omega t\ \hat{\imath} - R\omega^2\sin\omega t\ \hat{\jmath}$

Re-express (3) using (1): $\vec{a}(t) = -\omega^2\,\vec{r}(t)$

This proves that the acceleration vector points to the centre (opposite to $\vec{r}(t)$) ← in uniform circular motion

- Now calculate the speed from (2):

$$\underline{\underline{v(t)}} = |\vec{v}(t)| = \sqrt{v_x^2(t) + v_y^2(t)}$$

$$= \sqrt{(-R\omega\sin\omega t)^2 + (R\omega\cos\omega t)^2}$$

$$= \sqrt{R^2\omega^2\underbrace{(\sin^2\omega t + \cos^2\omega t)}_{=1\ \text{(Pythagorean in unit circle)}}} = \underline{\underline{R\omega}}$$

- does not depend on time!

(4) $v = R\omega = R\,2\pi f = R\frac{2\pi}{T}$. $\rightarrow$ $\omega = \frac{v}{R}$

- Acceleration magnitude:

$$a(t) = |\vec{a}(t)| = \sqrt{a_x^2(t) + a_y^2(t)} = |-\omega^2\,\vec{r}(t)|$$

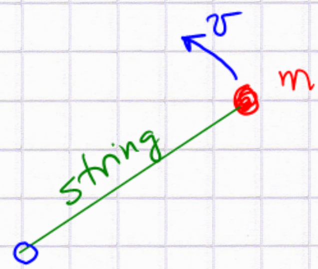
- also time-independent! $= \omega^2 R$

Derived (5) $a_{cp} = \omega^2 R$ $\xrightarrow{\text{use (4)}}$ $\boxed{a_{cp} = \frac{v^2}{R^2}\cdot R = \frac{v^2}{R}}$

Summary We describe circular uniform (constant speed) motion using an inertial reference frame.

Dynamics: By Newton-2 $m\vec{a} = \vec{F}$ we need a net force to provide the centripetal acceleration

Examples:

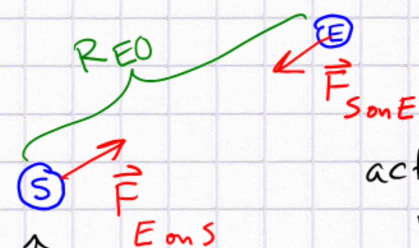


(1) mass m attached to a string is being hurled around $\longrightarrow$ string tension is providing

$$T = m a_{cp} = m \frac{v^2}{R}$$

$\longrightarrow$ smaller R $\rightarrow$ bigger T required

larger v (or ω) $\longrightarrow$ bigger T required



(2) earth going on near-circular orbit about sun

action-reaction pair

so massive ($M_S \gg M_E$), assume it's not moving (tiny acceleration from $\vec{F}_{E\,on\,S}$)

Law of Gravity

$$F_{S\,on\,E} = F_{E\,on\,S} = G \frac{M_E M_S}{R_{EO}^2}$$

$\vec{F}_{S\,on\,E}$ provides the centripetal acceleration to keep earth in a (near) circular (elliptic) orbit

$G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$,

$M_E \approx 6 \times 10^{24}$ kg,

$M_S \approx 2 \times 10^{30}$ kg, $R_{EO} = 1.5 \times 10^{11}$ m