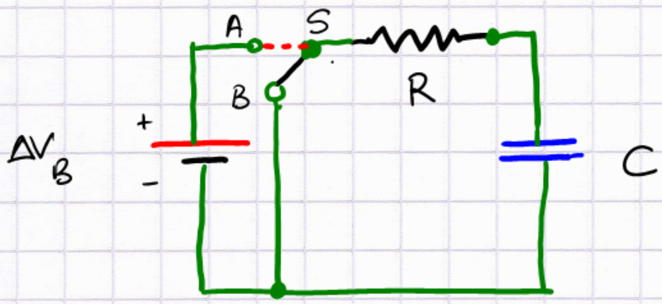
RC circuit

Define a circuit where a battery first charges a capacitor while the current passes through a resistor, and then we discharge the capacitor.



The switch can connect either

S to A (battery charges C via R)

or S to B (battery disconnected, C discharges via R)

We start with the S-B connection, there should be no charge on C, i.e., $Q = 0$. Now connect S-A.

A current will flow. Why?

potential drops $\Rightarrow$ "-"

Kirchhoff loop rule: $\Delta V_B - RI - \frac{Q}{C} = 0$

but at $t=0$ $Q=0$, i.e.,

$$\Delta V_B = RI$$

$$I(t=0) = \frac{\Delta V_B}{R}$$

At time $t=0$, when $Q(t=0)=0$, current flows through C without any opposition (amazing, given that there is a gap, but we are displacing charge!)

The current is only limited by the resistor R (Ohm's law)

However, for $t>0$ charge $Q(t)$ builds up on the capacitor plates $\Rightarrow$ $|\Delta V_C(t)| = \frac{Q(t)}{C}$

$\Rightarrow$ $|\Delta V_R| < |\Delta V_B|$ $\Rightarrow$ $I(t) = \frac{1}{R}\left(\Delta V_B - \frac{Q(t)}{C}\right)$

Q: can we figure out the function $I(t)$?

use Calculus: $I(t) = \frac{dQ}{dt}$ use this to get an eqn:

$$\frac{dQ}{dt} = \frac{\Delta V_B}{R} - \frac{1}{RC} Q(t)$$

define $\tau = RC$
(a time constant)

why does this follow?

$$\left[\frac{dQ}{dt}\right] = \frac{\text{charge}}{\text{time}} \Leftrightarrow \frac{Q(t)}{RC}$$

remember: $\frac{\Delta V_B}{R} = I_0$ initial current

$$\boxed{\begin{array}{l} \frac{dQ}{dt} = I_0 - \frac{Q(t)}{\tau} \\ Q(t=0) = 0 \quad \Rightarrow \text{initial condition} \end{array}}$$

which function $Q(t)$ solves that? It should involve an exponential function

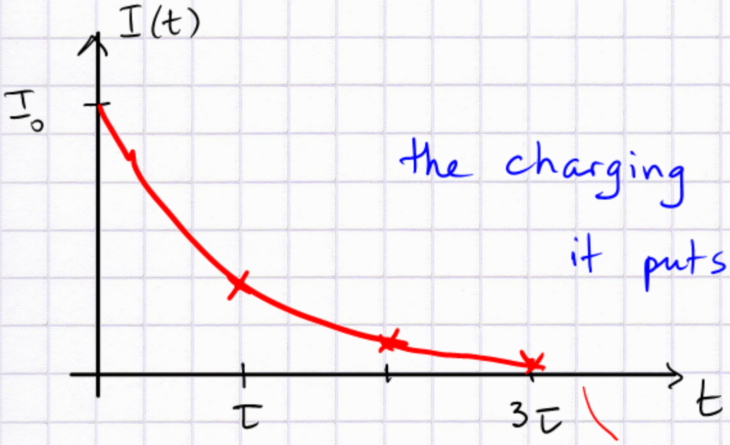
why? $\frac{dQ}{dt} \sim Q(t)/\tau$

can we simplify things?

Let's write an eqn for the current by taking the derivative:

$$\frac{d^2Q}{dt^2} = -\frac{1}{\tau}\frac{dQ}{dt} \Rightarrow \frac{dI}{dt} = -\frac{1}{\tau} I(t)$$

$$\therefore \quad I(t) = I_0 \, e^{-t/\tau}$$



t-axis = asymptote

the charging current dies exponentially:

it puts charge $Q(t)$ on the plates

↳ area under $I(t)$ curve

and $\Delta V_C = Q(t)/C$ builds up!

What does $\Delta V_C(t)$ look like?

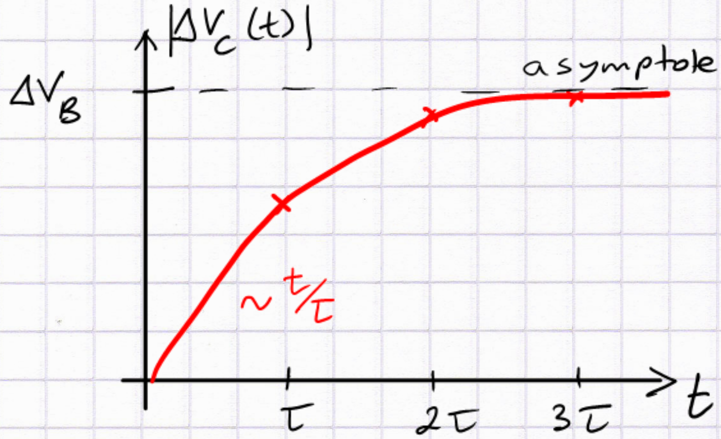
$$\Delta V_B + \Delta V_R + \Delta V_C = 0$$

Get this from loop law: $\Delta V_C = -\Delta V_B + R\,I(t)$

$$|\Delta V_C(t)| = \left|\Delta V_B\left(-1 + \frac{\cancel{R}\, I_0\, e^{-t/\tau}}{\cancel{\Delta V_B}}\right)\right| = \Delta V_B\left(1 - e^{-t/\tau}\right)$$

To understand use Taylor expansion $e^{-t/\tau} \simeq 1 - \frac{t}{\tau} + \frac{1}{2}\left(\frac{t}{\tau}\right)^2 \mp \cdots$

$$|\Delta V_C(t)| \underset{\text{small } t}{\sim} \Delta V_B \frac{t}{\tau} = \frac{\Delta V_B}{RC} t = \frac{1}{C} I_0 t$$



The discharge cycle:

a similar analysis,

but no battery

$$\Delta V_C(t) + \Delta V_R(t) = 0$$

$$-\frac{Q(t)}{C} - R\,I(t) = 0 \quad \therefore \quad -Q(t) = \tau\, I(t)$$

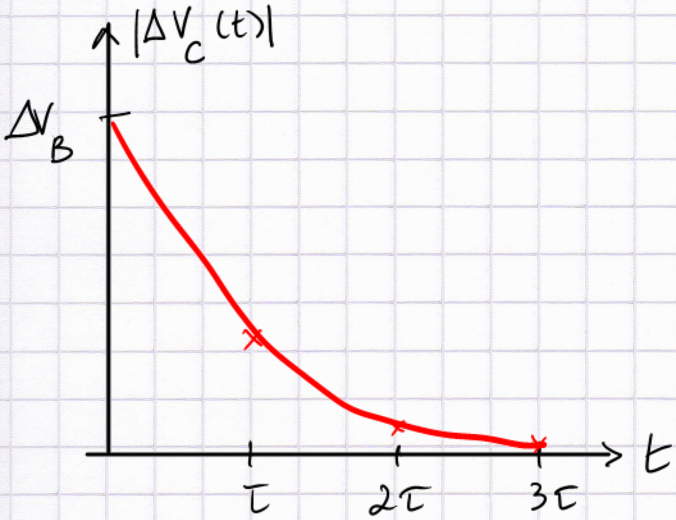
$$-\frac{Q(t)}{\tau} = I(t) = \frac{dQ}{dt}$$

$$Q(t) = Q_0\, e^{-t/\tau}$$

$$|\Delta V_C(t)| = \frac{Q_0}{C} e^{-t/\tau}$$

$$Q_0 = C\,\Delta V_B \quad \text{(when fully charged)}$$

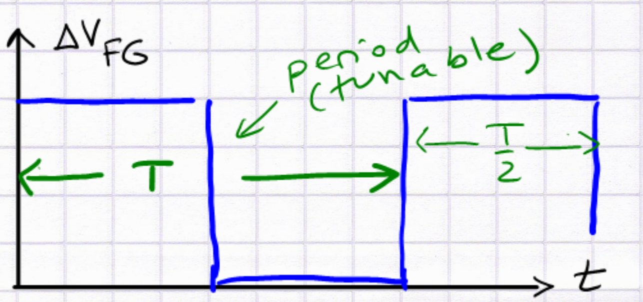
$$|\Delta V_C(t)| = \Delta V_B\, e^{-t/\tau}$$



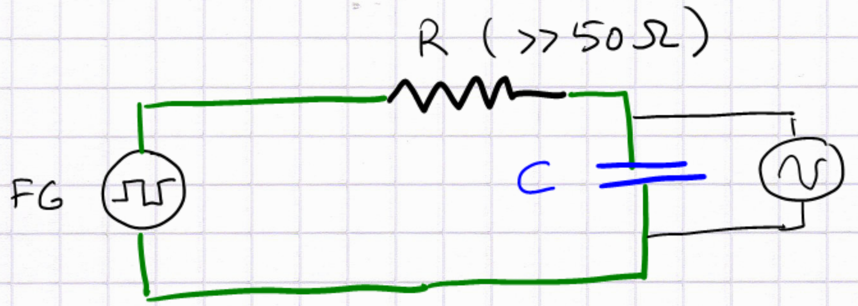
A function generator delivering

a square-wave pulse:

(50-50 duty cycle = symmetric ON-OFF)



The function generator acts to mimick the "battery + switch" configuration. When the generator is low it acts as a "short" (really a 50Ω resistor).



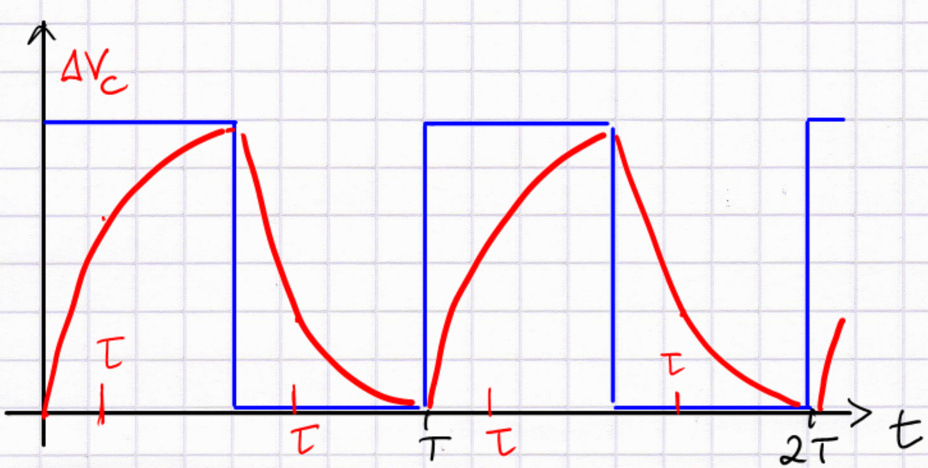
oscilloscope (channel 2)
(channel 1 directly connected to FG, which also triggers the scope)

The oscilloscope records the voltage drop across the charging / discharging capacitor periodically in time

The time base of the oscilloscope is adjusted to show 1 - 3 periods of the function generator T

The period of the FG is kept at $T >> \tau$ so that the charge / discharge cycles are nearly complete

The response curve shown in red is found in many places, e.g., in fMRI brain scans of neuronal activity in response to periodic "on-off" stimulus such as finger tapping.



Channel 1 displays the FG output
$T/2 \approx 3\tau$ was chosen

channel 2 shows the capacitor voltage

time base adjusted to span $2T$