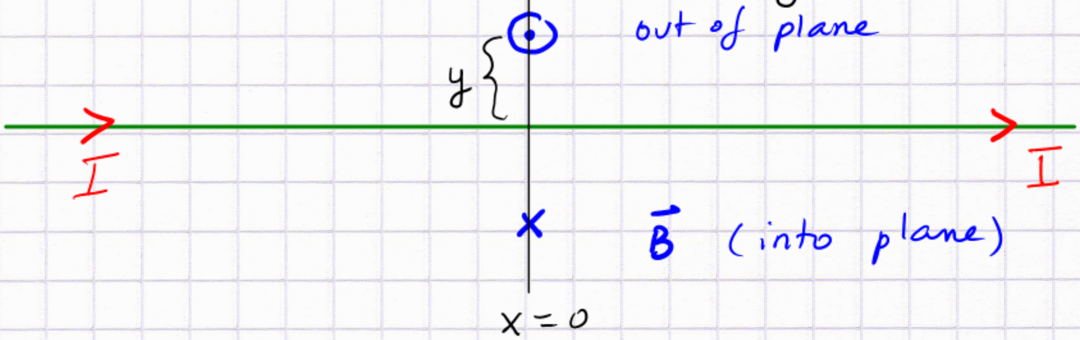
Biot - Savart : obtaining the $\vec{B}$ field for a current

Objective: given a straight, very long (infinite) wire with an electric current $\rightarrow$ obtain the B field strength as a function of the distance from the wire

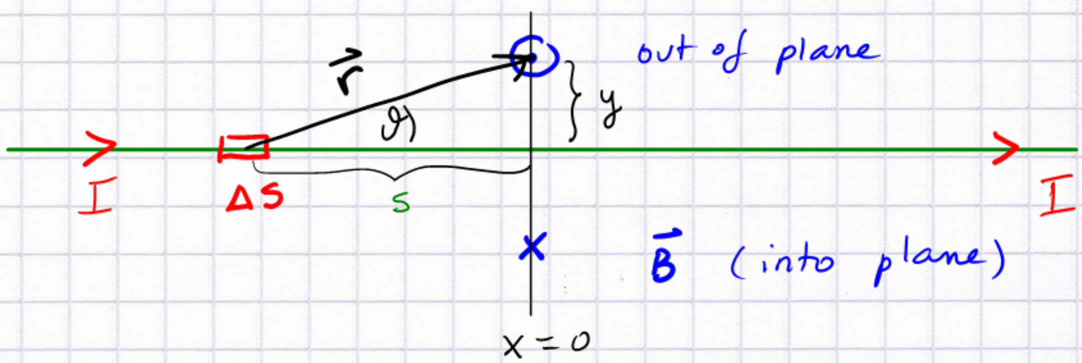


Start with some symmetries.

1) The $\vec{B}$ field lines are circles surrounding the wire $\therefore$ $B_x = 0$, $B_y = 0$, only $B_z \neq 0$ for the x-y plane locations

2) we need to calculate only for $x = 0$: for an ∞-ly long wire there should be no x-dependence of $B_z(x, y)$ $\therefore$ only $B_z(y)$ to be determined

3) $\vec{B} = \dfrac{\mu_0}{4\pi} \dfrac{I(\Delta\vec{s} \times \hat{r})}{r^2}$ requires us to chop the wire into small segments $\Delta\vec{s} = \hat{\imath}\,\Delta s$ and sum contributions



contribution from Δs at $(x = 0, y)$:

$$\Delta B = \frac{\mu_0}{4\pi} I \frac{\Delta s \cdot 1 \cdot \sin\vartheta}{r^2} = \frac{\mu_0}{4\pi} I \frac{\Delta s\, y}{r^3} = \frac{\mu_0}{4\pi} I \frac{\Delta s\, y}{(s^2 + y^2)^{3/2}}$$

Remarks, before we proceed to sum all ΔB:

- we expect the biggest contribution from $s=0$, since the denominator $\sim s^3$ (for small distance y from the wire)
- the Biot-Savart law is a "cousin" to Coulomb's law it has a $\frac{1}{r^2}$ fall-off built in, but is more complicated, as the current direction matters. However, if the point where we are calculating the $\vec{B}$ field $(x=0, y, z=0)$ was co-moving with the charge travelling inside the wire, then we would calculate an electric $\vec{E}$ field from static charge segments. $\vec{E}$ and $\vec{B}$ somehow transform into each other when changing reference frames $\longrightarrow$ understand in upper-year courses!

Total B field $B_z(x=0, y, z=0)$:

$$B_z = \sum \Delta B = \frac{\mu_0}{4\pi} I \boxed{\int_{-\infty}^{\infty} \frac{y\, ds}{(s^2+y^2)^{3/2}}}$$

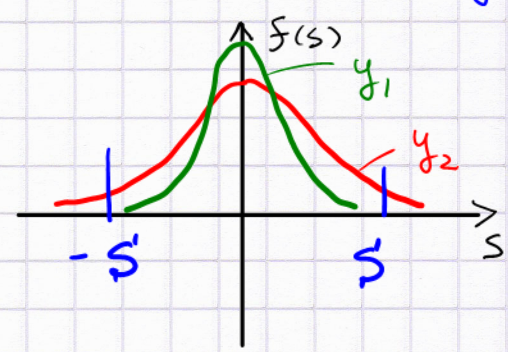
area under a bell-shaped curve which depends on the value of y

Calculating this area involves a double limit:

1) finite S': $\int_{-S'}^{S'} \frac{y\, ds}{(s^2+y^2)^{3/2}}$

2) then $\lim_{S' \to \infty}$

together: "improper integral" $\longrightarrow$



anti-derivative of $\frac{1}{(x^2+a^2)^{3/2}}$ is $\frac{x}{a^2\sqrt{x^2+a^2}}$

$$\therefore\ y\int_{-S'}^{S'} \frac{ds}{(s^2+y^2)^{3/2}} = y\left[\frac{S'}{y^2\sqrt{s^2+y^2}} - \frac{-S'}{y^2\sqrt{s^2+y^2}}\right]$$

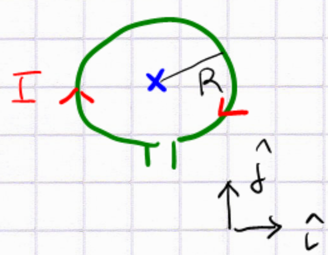
$$= \frac{2S'}{y\sqrt{S'^2+y^2}} \xrightarrow[S' \to \infty]{} \frac{2}{y}$$

$$\therefore\ \boxed{B_z(y) = \frac{\mu_0}{4\pi} I \cdot \frac{2}{y}}$$

@ $x=0, z=0$

Main result: a long, current-carrying wire surrounds itself with a $\vec{B}$ field, the field lines form circles, with direction given by the RH rule and a strength given by $\frac{\mu_0}{2\pi} I \left(\frac{1}{d}\right)$ where d is the perpendicular distance from the wire.

Analogous calculation for a loop of wire (radius R) leads to a formula for the field strength at the centre of the loop



$B_{z,\text{centre}} < 0$
(into the plane)
by RH rule

$$|B_z| = \frac{\mu_0}{2} \frac{I}{R}$$

For an N-turn coil (coil means: length << radius R):

$$|B_{z,\text{centre}}| = \frac{\mu_0}{2} \frac{N \cdot I}{R}$$

This is a practical result. Note: this is for an empty coil (no "soft-iron" core). An iron core can be used to strengthen the field. (iron is magnetized, but not permanently)

Units: B-field SI unit: Tesla $= \frac{1\,N}{A\,m}$

$$\mu_0 \equiv 4\pi \times 10^{-7} \frac{Tm}{A} = 1.257 \times 10^{-6} \frac{Tm}{A}$$

Example: coil of radius R=0.1 m, current 1.0 A, one turn (N=1)

$$B_{\text{centre}} = \frac{\mu_0}{2} \frac{I}{R} = \frac{1.26 \times 10^{-6}}{2} \frac{1.0}{0.1} \frac{Tm}{A} \frac{A}{m} = 6.5 \times 10^{-6}\,T$$

The earth's magnetic field at our latitude and @ surface

$\sim 50\,\mu T = 5.0 \times 10^{-5}\,T$

Thus, our simple current loop is weaker than the earth's magnetic field

$\Rightarrow$ in practice: use more turns

Solenoid vs coil: If the windings are put together such that the length of the electromagnet $L > R$, we call it a solenoid. A different calculation is needed, usually based on Ampère's law. (section 20.7)

On p. 665:

$$B_{sol.} = \frac{\mu_0 N I}{L}$$

A very similar formula, but the diameter $2R$ is replaced by the length L. Very puzzling!

A more realistic calculation would reveal that the true answer should depend on both R and L, perhaps replacing $2R$ and L by $\sqrt{L^2+4R^2}$ such that the limits $R \ll L$ and $R \gg L$ are both recovered.