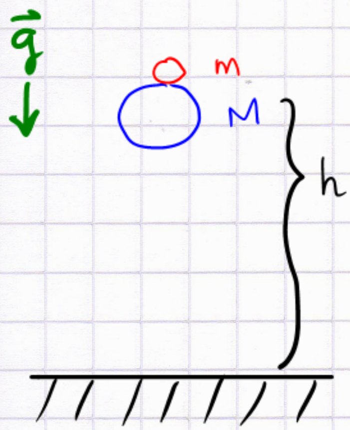
"Astrobounce"

A pair of balls, m sits on top of M, is dropped from height h. What happens?



A sequence of "events"

1) M moves from h to ground (= BOTTOM)
   m " " " " almost ground (= BOTTOM)

$$\underbrace{v_B = V_B}_{\text{speeds } (>0)} = \sqrt{2gh}$$

convert
$Mgh \to \frac{1}{2} M V_B^2$
$mgh \to \frac{1}{2} m v_B^2$

2) M (at the bottom) compresses, converts $KE = \frac{1}{2} M V_f^2 \to$ spring-like energy $\to$ reverses direction while m is still moving down $\to$ collision

Detailed analysis: make a $\hat{j}$ choice, e.g., $\uparrow \hat{j}$, then:

$$M V_B - m v_B = M V_f + m v_f$$

reversed, going up — coming down!

V_f, v_f are velocities (with sign) after collision

speeds at bottom $\longrightarrow V_B = v_B = \sqrt{2gh}$ (>0)

$$(M-m)\sqrt{2gh} = M V_f + m v_f \quad \to \quad V_f = (M-m)\sqrt{2gh} - m v_f$$

A) One equation, two unknowns (V_f, v_f) cannot solve

B) Why was momentum conservation valid? The system is NOT force free! Gravity + normal force are external forces acting on M, gravity acts on m.

Answer: collision takes place over short times ($\Delta t \sim$ msec)
External forces cannot change p^{tot} by much over Δt

C) Energy conservation during (perfectly) elastic collision:

$$\frac{1}{2} M (2gh) + \frac{1}{2} m (2gh) = \frac{1}{2} M V_f^2 + \frac{1}{2} m v_f^2$$

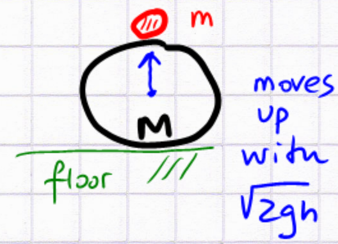
eliminate V_f + solve for v_f
TEDIOUS! doable!

Simplified analysis for the M-ball + m-ball collision :

"back of envelope"

go to the reference frame moving with M :

in this frame : m comes down with $2\sqrt{2gh}$:



$$M \cdot 0 - m\left(2\sqrt{2gh}\right) = M\tilde{V}_f + m\tilde{v}_f$$

$$\therefore \quad \tilde{V}_f = -\frac{m}{M}\left(2\sqrt{2gh}\right) - \frac{m}{M}\tilde{v}_f$$

In this frame M will see a (small) recoil (downward velocity, $\frac{m}{M}$ factor → small)

$M \gg m$ analysis : in this frame, m "bounces" from a wall

→ reverses its velocity → $\tilde{v}_f \approx 2\sqrt{2gh}$ up

With respect to the floor : $v_f = \tilde{v}_f + v_{frame} = 3\sqrt{2gh}$
(absolute space)

$\uparrow$ frame: M moves up with $\sqrt{2gh}$!

$\therefore$ In $M > m$ limit : top ball goes from

$\sqrt{2gh}$ down → $3\sqrt{2gh}$ up

Now convert KE → PE for top ball :

$$\cancel{m}gH = \frac{1}{2}m v_f^2 = \frac{1}{2}\cancel{m}\cdot 9 \cdot 2\cancel{g}h$$

$$H = 9h$$ nine-fold original height !

Physics message :
- collisions allow transfer of momentum
- this also implies transfer of KE

- big ball has a lot more energy : Mgh vs mgh ($M \gg m$)
- transfers a big part of Mgh (almost comes to rest @ BOTTOM)
- top ball gains this energy; theoretically could reach up to $H = 9h$.

Appendix Do the full calculation in the "normal" frame and then look at $M \gg m$.

$$(M-m)\sqrt{2gh} = MV_f + m v_f \quad (1)$$

total momentum conservation

$$\frac{1}{2}M(2gh) + \frac{1}{2}m(2gh) = \frac{1}{2}MV_f^2 + \frac{1}{2}m v_f^2 \quad (2)$$

energy conservation during elastic collision

How do we solve for v_f and V_f ?

Use (1) to eliminate V_f in (2):

$$V_f = \left(1-\frac{m}{M}\right)\sqrt{2gh} - \frac{m}{M}v_f$$

$$(2): \quad 2gh(M+m) = MV_f^2 + m v_f^2$$

$$\therefore \quad 2gh(M+m) = M\left(\left(1-\frac{m}{M}\right)\sqrt{2gh} - \frac{m}{M}v_f\right)^2 + m v_f^2$$

Sort out the mess $\longrightarrow$ quadratic eqn in v_f ...

$$2gh\,\frac{m+3M}{m+M} = v_f^2 - 2\sqrt{2gh}\,\frac{M-m}{M+m}\,v_f$$

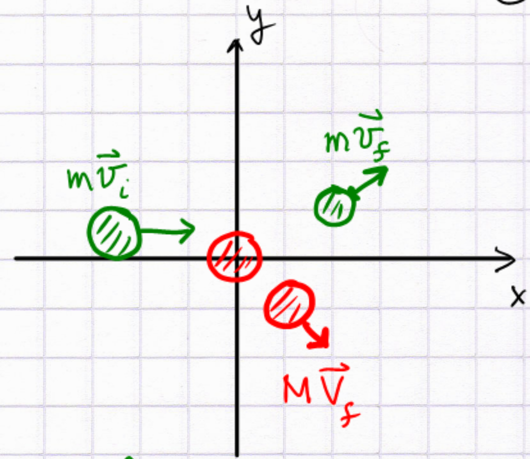
$$v_f = \sqrt{2gh}\,\frac{M-m}{M+m} \pm \sqrt{2gh\left(\frac{M-m}{M+m}\right)^2 + 2gh\,\frac{3M+m}{M+m}}$$

$$= \underset{v_B\,!}{\sqrt{2gh}}\left[\underbrace{\frac{M-m}{M+m}}_{\approx 1 \text{ for } M\gg m} + \sqrt{\frac{(M-m)^2 + (3M+m)(M+m)}{(M+m)^2}}\right]$$

increase!

$$= \frac{v_B}{M+m}\left[M-m+\sqrt{\underbrace{M^2-2mM+m^2+3M^2+4mM+m^2}_{4M^2+2mM+2m^2}}\right] \rightarrow \frac{M+2M}{M}v_B \xrightarrow[M\gg m]{} 3v_B$$

Elastic Collisions in 2D



hockey pucks colliding off-center

initially : $M\vec{V}_i = \vec{0}$

$m\vec{v}_i = m v \hat{\imath}$

finally : $M\vec{V}_f$, $m\vec{v}_f \rightarrow m v_{fx}\hat{\imath} + m v_{fy}\hat{\jmath}$

$\longrightarrow M V_{fx}\hat{\imath} + M V_{fy}\hat{\jmath}$

Momentum conservation : vector eqn $\rightarrow$ 2 eqs

$$m v_{ix} + 0 = m v_{fx} + M V_{fx} \; ; \quad 0 = m v_{fy} + M V_{fy}$$

Energy conservation : one scalar eqn

$$\frac{1}{2} m v_{ix}^2 = \frac{1}{2} m \left(v_{fx}^2 + v_{fy}^2\right) + \frac{1}{2} M \left(V_{fx}^2 + V_{fy}^2\right)$$

Directional info : $v_{fx} = v_f \cos\vartheta$, $v_{fy} = v_f \sin\vartheta$

$V_{fx} = V_f \cos\varphi$, $V_{fy} = -V_f \sin\varphi$

Q : how would we show that $(\vartheta + \varphi) \approx \frac{\pi}{2}$ for peripheral collisions as demonstrated in the video ?

m=M case $\rightarrow$ $v \sin\vartheta = -V \sin\varphi$, $v_0 = v\cos\vartheta + V\cos\varphi$,

$v_0^2 = v^2 + V^2$;

Note : we have 3 equations. For given input momenta there are 4 unknowns : $(v, V, \vartheta, \varphi)$ or (p_x, p_y, P_x, P_y)

$\therefore$ we cannot determine the outcome uniquely !

We can discuss special cases.

1) Equal energy sharing $v = V$; since $m = M$ by assumption

$\frac{1}{2} m v^2 = \frac{1}{2} M V^2$

What follows?

$v_0^2 = v^2 + V^2 = 2v^2 \quad \therefore \quad v = \frac{v_0}{\sqrt{2}}$

$v \sin\vartheta = -V \sin\varphi \quad \therefore \quad v \sin\vartheta = -v \sin\varphi$ | but $v \neq 0$

$\therefore \quad \vartheta = -\varphi$

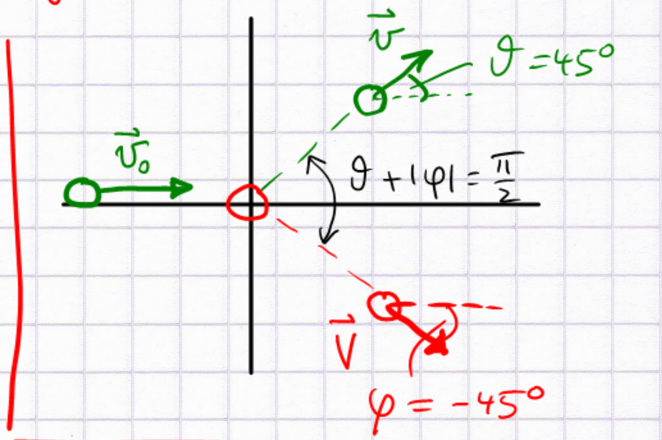
Now use: $v_0 = v\cos\vartheta + v\cos\varphi = v(\cos\vartheta + \cos(-\vartheta))$

$v_0 \rightarrow \sqrt{2}\, v \qquad = 2v\cos\vartheta$

$\therefore \quad \cos\vartheta = \frac{1}{2}\sqrt{2} \quad \therefore \quad \vartheta = \frac{\pi}{4}, \ \varphi = -\frac{\pi}{4}$

Note: $\vartheta + |\varphi| = \frac{\pi}{2}$

Equal-mass pucks collide: m hits stationary M → for both to come out with equal energy they emerge as



2) A grazing collision:

small ϑ (little deflection)

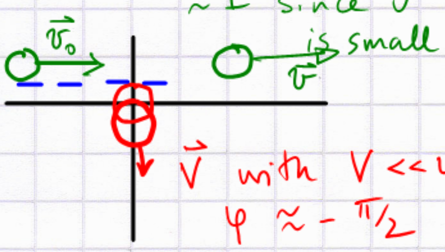
expect $v \approx v_0$, $V \ll v_0$.

(from $v_0^2 = v^2 + V^2$)

$P_y = m v \sin\vartheta = -P_y = -MV\sin\varphi$

if $\sin\vartheta$ is small & v is large $\therefore$ if V is small, $\sin\varphi$ is large

$v_0 = v \underbrace{\cos\vartheta}_{\approx 1 \text{ since } \vartheta \text{ is small}} + V\cos\varphi \quad \therefore \quad \cos\varphi = \frac{v_0 - v}{V} \quad \frac{\text{small}}{\text{small}}$?



How can we make the argument tight?

$v_0^2 - v^2 = (v_0 - v)(v_0 + v) = V^2$

$\therefore \quad v_0 - v \simeq \frac{V^2}{2v_0} \quad \therefore \quad \cos\varphi \simeq \frac{V}{2v_0}$ really small $\rightarrow \varphi = -\frac{\pi}{2}$

again: $\vartheta + |\varphi| \approx \frac{\pi}{2}$.