Moment of Inertia

For rotations about a fixed axis we derived previously the analogy to 1d linear motion:

$m \rightarrow I \sim mR^2$ → single mass pt a distance R away from rotation axis

↘ $\gamma m R^2$ → γ = geometric factor

$a \rightarrow \alpha \; (=\alpha_z)$ linear vs angular acc.

$v \rightarrow \omega \; (=\omega_z)$ linear vs angular velocity

$x \rightarrow \vartheta \; (=\vartheta_z)$ linear vs angular position

$F_x^{net} \rightarrow \tau_z^{net}$ net force → net torque about rot. axis

$$m a_x = F_x \longrightarrow I\alpha_z = \tau_z \qquad (I = I_{zz})$$

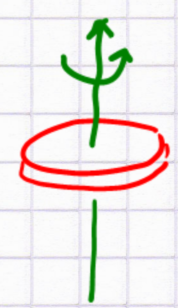
rotational inertia about z-axis

Some more insight about I = rot. inertia:

$\vec{\omega} = (0, 0, \omega_z)$

Consider a disk, rotation axis through CM

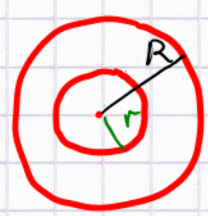
R = radius, m = mass, thin (coin)



RH rule: $\omega_z > 0$ CCW rot.

How is the mass distributed?

top view:



Disk is made up of rings of radius r, thickness Δr; $r = 0..R$

thin disk (ignore height): surface mass density $\sigma = \frac{m}{\pi R^2}$

= total mass / total area

total mass: $m = \sum_{\Delta A} \sigma \Delta A$ sum the masses of rings which make up the disks

Q: How much area within each ring of radius r ?

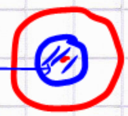
$\longrightarrow \quad 2\pi r \Delta r$

$$\therefore \quad m = \sum_{\Delta r} \underbrace{\frac{m}{\pi R^2}}_{\sigma} (2\pi r \Delta r) = \frac{2m}{R^2} \sum_{\Delta r} r \Delta r \longrightarrow m \quad ✓$$

$$\int_0^R r\,dr = \frac{1}{2} r^2 \Big|_0^R = \frac{1}{2} R^2$$

What is this calculation good for?

Q: how much mass is contained in the coin up to half the radius?



$$m_{(R/2)} = \frac{2m}{R^2} \int_0^{R/2} r\,dr = \frac{2m}{R^2} \cdot \frac{1}{2} r^2 \Big|_0^{R/2} = \frac{m}{R^2}\left(\frac{R^2}{4}\right) = \frac{m}{4}.$$

This wasn't obvious? Of course, just use πR^2 formula for the area with $R \to \frac{R}{2}$!

So, what is this calc. really good for?

Inertia calc. $\longrightarrow$ each ring of mass $dm = \sigma dA$

or: $\Delta m = \sigma (2\pi r \Delta r) = \frac{m}{\pi R^2}(2\pi r \Delta r) = \frac{2mr}{R^2} \Delta r$

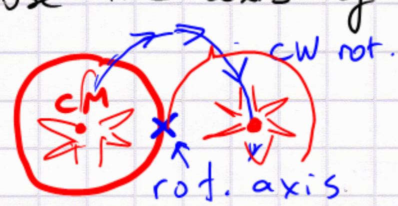
contributes with $r^2 \Delta m$ to I_{tot}:

$$I = \int r^2 dm = \sum_{\Delta m} r^2 \Delta m = \sum_{\Delta r} r^2 \frac{2mr}{R^2} \Delta r = \frac{2m}{R^2} \sum_{\Delta r} r^3 \Delta r$$

$$= \frac{2m}{R^2} \int_0^R r^3 dr = \frac{2m}{R^2} \times \frac{r^4}{4}\Big|_0^R = \frac{2m}{R^2}\frac{R^4}{4} = \frac{1}{2} mR^2$$

We derived the moment of inertia of a disk about its CM

Suppose the axis of rotation was moved to the rim:



$I_0 = I_{CM} + mR^2 = \frac{3}{2} mR^2$

why? all of m (= CM) rotates about X

$= mR^2$ + coin spins once around about CM