Wave properties: superposition and interference

Wave pulse = mixture of infinite single-frequency/wavelength trains

$\Rightarrow$ study $\quad D(x,t) = A \sin\left(\frac{2\pi}{T} t - \frac{2\pi}{\lambda} x\right)$

Observe : $\quad \frac{d^2}{dx^2} D(x,t) = -A\left(-\frac{2\pi}{\lambda}\right)\left(-\frac{2\pi}{\lambda}\right) \sin\left(\frac{2\pi}{T} t - \frac{2\pi}{\lambda} x\right)$

$$= -\left(\frac{2\pi}{\lambda}\right)^2 D(x,t)$$

$$\frac{d^2}{dt^2} D(x,t) = -A\left(\frac{2\pi}{T}\right)\left(\frac{2\pi}{T}\right) \sin\left(\frac{2\pi}{T} t - \frac{2\pi}{\lambda} x\right)$$

$$= -\left(\frac{2\pi}{T}\right)^2 D(x,t)$$

Combine this info into the so-called wave equation:

$$\frac{\partial^2}{\partial t^2} D(x,t) \sim \frac{\partial^2}{\partial x^2} D(x,t)$$

from isolating + equating $D(x,t)$ in the above 2 statements

1) use $\frac{\partial}{\partial x}, \frac{\partial}{\partial t}$ symbols instead of $\frac{d}{dx}, \frac{d}{dt}$

to recognize: (x,t) are independent variables,

unlike in Class mech, where $x = x(t) \rightarrow$ trajectory

2) on dimensional grounds if we wish to

turn the proportionality into an equation $\rightarrow$

we need a factor involving $\frac{[\text{Length}]^2}{[\text{Time}]^2} = [\text{velocity}]^2$

$$D(x,t) = -\left(\frac{T}{2\pi}\right)^2 \frac{\partial^2}{\partial t^2} D(x,t) = -\left(\frac{\lambda}{2\pi}\right)^2 \frac{\partial^2}{\partial x^2} D(x,t)$$

$$\therefore \quad \frac{\partial^2}{\partial t^2} D(x,t) = \left(\frac{\lambda}{T}\right)^2 \frac{\partial^2}{\partial x^2} D(x,t)$$

$$\frac{\partial^2}{\partial t^2} D(x,t) = c^2 \frac{\partial^2}{\partial x^2} D(x,t) \quad \text{wave equation}$$

$c \equiv$ wave propagation speed is independent of $f = \frac{1}{T}$ or λ !!

Many waves satisfy this equation to high accuracy

→ waves which dissipate (change form, lose their mechanical energy → heat) do not satisfy it (over long times).

Fundamental waves in physics (electromagnetic waves, waves describing quantum particles), and some not so fundamental ones (e.g. longitudinal sound waves in a gas, or in a solid!) are described by laws of motion of this form

"rate of change in time" is related to "rate of change in space"

(a PDE or partial differential equation)

$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial t}$ are called partial derivatives.

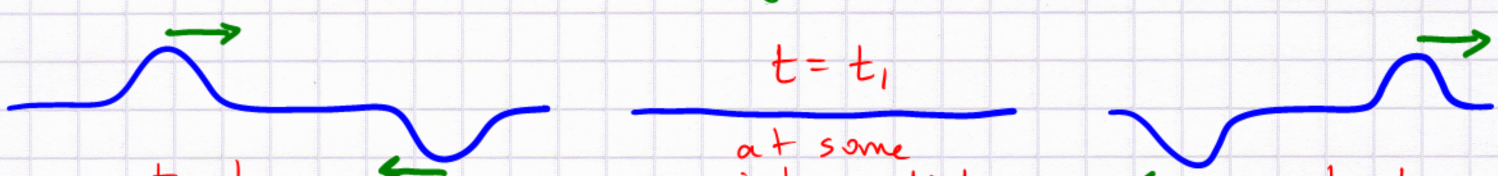
We investigate here the most basic properties, namely the relation between the spatial and temporal properties:

Q: How are wavelength λ and frequency f related? (beyond $\lambda \cdot f = c$)

→ allowed tones on a string?

→ fundamental vibration and overtones

Superposition: Imagine two pulses counterpropagating on a rope (string):



$t = t_0$ | $t = t_1$ at some intermediate time the pulses cancel! | $t = t_2$

A linear wave equation allows such solutions.

It allows many other things related to the additivity of wave solutions, such as the addition of crests + troughs
$\rightarrow$ wave interference phenomena

Interlude: Mathematics of Interference
= Addition Theorem for sin / cos

Q: what happens when a left- and right-travelling sinusoidal wave of the same frequency meet?

Assume $v = v_0 > 0$ $L \to R$

$$D_1(x,t) = A \sin\left(\frac{2\pi x}{\lambda} - \frac{2\pi t}{T}\right) \qquad \frac{+\lambda}{T} = v_0 \ (=c)$$

$R \to L$ wave:

$$D_2(x,t) = A \sin\left(\frac{2\pi x}{\lambda} + \frac{2\pi t}{T}\right)$$

Why? $\lambda \to -\lambda$ in above $\rightarrow -v_0$

Now add: $D_{comb}(x,t) = D_1(x,t) + D_2(x,t)$

$$= A\left[\sin\left(\frac{2\pi x}{\lambda} - \frac{2\pi t}{T}\right) + \sin\left(\frac{2\pi x}{\lambda} + \frac{2\pi t}{T}\right)\right]$$

Is there a trigonometry result for $\sin\alpha + \sin\beta$?

You probably (should) know: $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

Set $\alpha = \frac{2\pi x}{\lambda}$ $\beta = \frac{2\pi t}{T}$

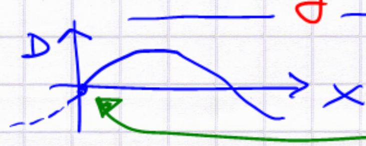
$$\frac{1}{A} D_{comb} = \sin(\alpha+\beta) + \sin(\alpha-\beta) = \sin\alpha\cos\beta + \cancel{\cos\alpha\sin\beta} + \sin\alpha\cos\beta - \cancel{\cos\alpha\sin\beta}$$

$$= 2\sin\alpha\cos\beta = 2\sin\left(\frac{2\pi x}{\lambda}\right)\cos\left(\frac{2\pi t}{T}\right)$$

we like this solution: $D(0,t) = 0$
= fixed string at 0

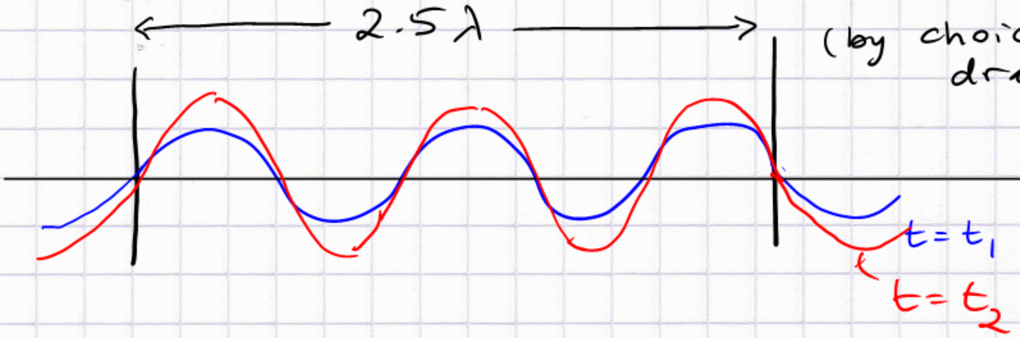
Therefore, $D_{comb} = 2A \underbrace{\sin\left(\frac{2\pi x}{\lambda}\right)} \underbrace{\cos\left(\frac{2\pi t}{T}\right)}$ is a standing wave

which oscillates in time



Remarkable result : 

creates a spatial structure which does not move :



(by choice of drawing)

Mathematically : $D_{comb}(x,t) = \underbrace{2A\cos\left(\frac{2\pi t}{T}\right)}_{\text{time dependent amplitude "modulates" the}} \cdot \underbrace{\sin\left(\frac{2\pi x}{\lambda}\right)}_{\text{spatial structure (doesn't change with time)}}$

$\tilde{A}(t) = 2A\cos\left(\frac{2\pi t}{T}\right)$

This is exactly what we need to describe what a string fixed at both ends is allowed to do.

The solution shown exists for all x, but we just look within some range $0 \leq x \leq L$: for $x=0$ and $x=L$ the displacement $D(0,t) = D(L,t) = 0$ for all t. (the string is fixed, and has length L).

Q : what are the allowed spatial structures ?

A : whenever $\sin\left(\frac{2\pi L}{\lambda}\right) = 0$ (fixed right endpoint, the left was fixed by $x=0$)

This has ∞-ly many solutions :

$n=1$: $\frac{2\pi L}{\lambda} = \pi \rightarrow \lambda_1 = 2L$

$n=1,2,3,\ldots$: $\boxed{\lambda_n = \frac{2L}{n}}$ allowed wavelengths!

$n=2$: $\frac{2\pi L}{\lambda} = 2\pi \rightarrow \lambda_2 = L$

$n=3$: $\frac{2\pi L}{\lambda} = 3\pi \rightarrow \lambda_3 = \frac{2L}{3}$

Associated frequencies: $\boxed{f_n = \frac{v_0}{\lambda_n}}$

shorter wavelength $\longrightarrow$ higher frequency. v_0 is fixed $= \sqrt{F_T/\mu}$

Thus, a string (piano, harp, guitar, violin, etc.)
which has a well-known length L, which is tuned by adjusting the tension force, and which by design has a given mass density (high-pitch → thin wire)

- tension force F_T + mass density μ → propagation speed v_0 → relationship between λ_n and f_n! –

such a string can produce standing waves with an infinite set of discrete wavelengths / frequencies.

When we pluck a string (finger, hammer, bow) → we do not produce a pure single-frequency sound, but a mixture: $f_1 + f_2 + f_3 + \ldots$ with well-defined proportions.

Wind instruments (can) produce pure, single-frequency tones, although harmonics are possible

→ remember recorder practice in grade 3 + 4!