Angular Momentum

We learned from the translation table 1d linear motion $\leftrightarrow$ rotational motion about a fixed (z) axis

$m \leftrightarrow I$

$v_x \leftrightarrow \omega_z$

Thus $p_x = m v_x \leftrightarrow l_z = I \omega_z$

$KE = \frac{1}{2} m v_x^2 = \frac{1}{2} \frac{p_x^2}{m}$ $\qquad \frac{1}{2} I \omega_z^2 = \frac{1}{2} \frac{l_z^2}{I}$

What is $l_z = I\omega_z$ good for?

Linear momentum of a complex system: total momentum is conserved in the absence of external forces

$\therefore$ no external torques $\rightarrow$ $l_z = I\omega_z$ is conserved?

Example: ice skater uses legs + skate vs ice friction to spin up, then no more skate-ice interactions

$\rightarrow$ External torque $\approx \emptyset$. $\qquad L_z = I_{zz}\,\omega_z$ is constant.

Now translate total momentum conservation:

no external force:

$\Delta p_1 + \Delta p_2 = 0$

$\downarrow$

$\sum_{i=1}^{N} \Delta p_i = 0$

speculate: no outside torque

$\rightarrow \Delta L_z = 0$

$\Delta L_z^{tot} = \Delta\left(\sum m_i r_i^2\right)\omega_z = 0$

$L_z^{fin} = L_z^{in}$

figure skater spinning + changing inertia $\leftarrow$ $I_{zz}^{fin}\,\omega_z^{fin} = I_{zz}^{in}\,\omega_z^{in}$

spinning figure skater with arms stretched out

$\rightarrow \quad I^{in} \omega^{in} = L_z^{in}$

pulls in arms $\rightarrow$ inertia is reduced $\quad I^{fin} < I^{in}$

no substantial torques act during the change

$\rightarrow L_z^{fin} = L_z^{in} \quad \therefore \quad I^{fin}\omega^{fin} = I^{in}\omega^{in}$

$\omega^{fin} = \frac{I^{in}}{I^{fin}} \omega^{in} > \omega^{in}$

figure skater spins up $\quad \omega^{fin} > \omega^{in}$
without outside torques

Interesting observation: $\quad KE^{fin} > KE^{in}$ since:

$$KE^{fin} = \frac{1}{2} I^{fin} (\omega^{fin})^2 = \frac{1}{2} I^{fin} \left(\frac{I^{in}}{I^{fin}}\right)^2 (\omega^{in})^2$$

$$= \frac{I^{in}}{I^{fin}} KE^{in} > KE^{in}$$

Q: where does the energy change come from?

A: skater does work while pulling in the arms (+legs)
(centrifugal force pushes mass outwards)

Rotational dynamics

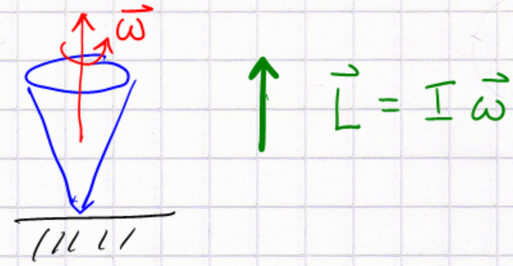
$$\frac{d}{dt} P_x = F_x \quad \longleftrightarrow \quad \frac{d}{dt} L_z = \tau_z$$

e.g. $I \frac{d}{dt} \omega_z = \tau_z \rightarrow I\alpha_z = \tau_z$

3d generalization: $\quad \frac{d}{dt}\vec{L} = \vec{\tau}$ $\quad \rightarrow \vec{L}$ - directional change $\quad \rightarrow |\vec{L}|$ - magnitude change

Examples

1) spinning vertical top



$$L_z = I\,\omega_z$$

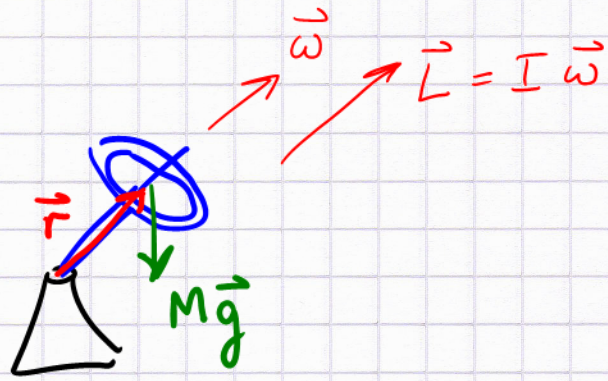
no gravitational torque $\longrightarrow$ rotation axis remains constant

$\longrightarrow$ rotation rate remains constant

with friction $\longrightarrow$ ω_z and L_z decrease over time

$$\frac{d}{dt} L_z < 0$$

2) tilted gyroscope



$$\vec{\tau} = \vec{r} \times (M\vec{g})$$

$$\frac{d}{dt}\vec{L} = \vec{\tau} \quad \therefore \quad \vec{L}_{t+\Delta t} = \vec{L}_t + \Delta t\,\vec{\tau}$$

Torque pushes the $\vec{L}$-axis around $\longrightarrow$ precession

gyroscope rotates slowly about the vertical axis

Gyroscope slows down (due to friction) $\longrightarrow$ precession rate increases

Non-spinning gyroscope ($L \approx 0$) $\longrightarrow$ torque makes it fall

$$\frac{d}{dt}\vec{L} = \vec{\tau}$$ angular momentum theorem

where does it come from ?

So far, we defined only $L_z = I\omega_z$ where I is the inertia for rotations about a fixed z axis

In general, the definition of angular momentum of a point particle is $\vec{L} \equiv \vec{r} \times \vec{p}$

Note: $L_z = rp\sin(\measuredangle\, \vec{r},\vec{p})$

We justify the angular momentum theorem by starting with Newton's 2^{nd} law:

$$\frac{d}{dt}(\vec{p}) = \vec{F}$$ | apply cross product with $\vec{r}$ from left

$$\vec{r} \times \frac{d}{dt}(\vec{p}) = \vec{r} \times \vec{F} = \vec{\tau}$$

RHS = what we want

LHS of ang. mom. th:

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \underbrace{\vec{r} \times \frac{d\vec{p}}{dt}}_{= \text{LHS!}}$$

(product rule)

We need to show: $\frac{d\vec{r}}{dt} \times \vec{p} \equiv 0$

Note: $\frac{d\vec{r}}{dt} = \vec{v}$

$\vec{p} = m\vec{v}$

$$\vec{v} \times (m\vec{v}) = m\,\vec{v} \times \vec{v} \equiv 0$$

Thus, we find that the ang. mom. theorem is a re-write of (one half of) Newton's 2^{nd} law!