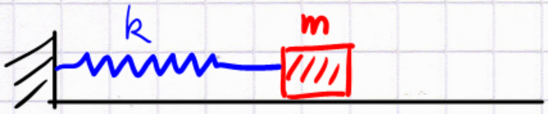
Harmonic motion



pull the mass from equilibrium + let it go → what happens?

X-coordinate system: choose $x = 0$ @ equilibrium point

$$\rightarrow \quad F_H(x) = -k\,x$$

why? $x > 0$: mass is pulled back to $x=0$

$x < 0$: mass is pushed towards $x=0$

Newton's 2nd law:

$$m \frac{d^2}{dt^2} x(t) = -k\,x(t) \quad \leftarrow \text{net force}$$

Q: which function(s) $x(t)$ satisfy this equation?

$$x''(t) = -\frac{k}{m}\,x(t)$$

idea: k/m will come from the chain rule

→ which function(s) satisfy $f'' = -f$?

$$x(t) = A\cos\omega t + B\sin\omega t$$

$$x'(t) = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$x''(t) = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t = -\omega^2 x(t)$$

∴ This works for any A, B as long as $\omega^2 = \frac{k}{m}$

$$\omega = \sqrt{\frac{k}{m}} = 2\pi f = \frac{2\pi}{T}$$

circular frequency of the oscillator

What is the meaning of A, B?

The period T, frequency $f = \frac{1}{T}$ are set by k and m

the physical characteristics of the oscillator.

A, B will be determined by how we set the system in motion:

→ give it a kick while it is in equilibrium

→ $x(0) = 0$, $v(0) = x'(0) = v_0$

↓

$A = 0$, $B \neq 0$

$x(t) = B \sin \omega t$, $x'(t) = v(t) = \underbrace{\omega B}_{v_0} \cos \omega t$

→ pull it from equilibrium $(x_0 \neq 0)$, + release $(v_0 = 0)$:

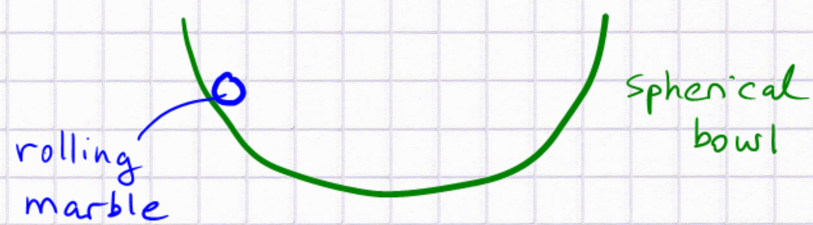
$A \neq 0$, $B = 0$

$x(t) = \underbrace{A}_{x_0} \cos \omega t$, $v(t) = \underbrace{-\omega A \sin \omega t}_{= 0 \ @ \ t = 0}$

→ most general case: $x_0 \neq 0$, $v_0 \neq 0$

pull from equilibrium + give it a kick.

Other oscillators:



both $x(t)$ ↔ and $y(t)$ ↕

represent oscillators

∴ springs are not the only devices which try to restore an object to a special location

Specialty of the simple oscillator: the frequency (period) DOES NOT DEPEND ON THE AMPLITUDE x_0 !

Potential and Kinetic Energies look similar:

$$PE = V_H = \frac{1}{2} k x^2 \qquad KE = \frac{1}{2} m v^2$$

Mechanical Energy is conserved. Consider the case where the mass is pulled from equilibrium and released:

$$x_0 \neq 0 \;, \quad v_0 = 0 \qquad E = \underbrace{\cancel{KE}}_{v_0 = 0} + PE \rightarrow \frac{1}{2} k x_0^2$$

Show that $E = \frac{1}{2} k x_0^2$ at all times:

$$x(t) = x_0 \cos \omega t \qquad v(t) = -\omega x_0 \sin \omega t$$

$$PE(t) = \frac{1}{2} k x_0^2 \cos^2 \omega t \qquad KE(t) = \frac{1}{2} m (\omega^2 x_0^2) \sin^2 \omega t$$

$$E(t) = KE(t) + PE(t) = \frac{1}{2} m \left(\frac{k}{m} x_0^2\right) \sin^2 \omega t + \frac{1}{2} k x_0^2 \cos^2 \omega t$$

$$= \frac{1}{2} k x_0^2 \underbrace{\left(\sin^2 \omega t + \cos^2 \omega t \right)}_{=1} = E$$

The simple oscillator goes on forever

(not very realistic → introduce damping)

Damped oscillator:

$$m \frac{d^2 x}{dt^2} = - k x(t) \underbrace{- b \frac{dx}{dt}}_{\text{opposes velocity}}$$