The pendulum

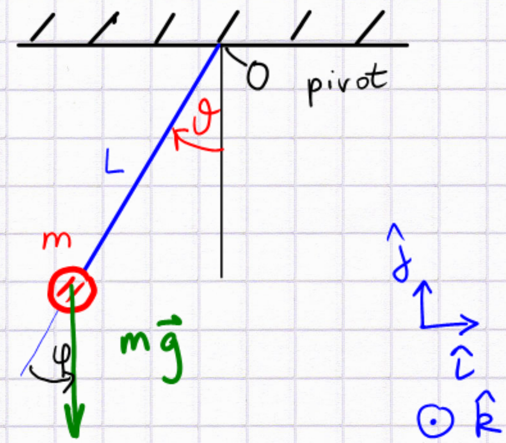
A fascinating oscillator: 2d motion (in x and y), gravity (rather than a spring) provides a restoring force.

Simple motion for small amplitudes ($\to$ harmonic oscillations), but not so simple when the arm is allowed to go near the top (motion slows down!), or even over the top, when the pendulum becomes a rotator.

We use rotational dynamics + discuss a "mathematical" pendulum:

massless rod of length L } inertia about pivot O: $I_0 = m L^2$

mass m



$$I_0 \alpha = I_0 \frac{d^2\vartheta}{dt^2} = \tau_z \leftarrow \text{gravitational torque about O}$$

$L m g \sin\varphi$ sign? $\varphi = -\vartheta$!

By the RH rule: $\tau_z > 0$

$$\underset{\text{thumb}}{\vec{L}} \times \underset{\text{index finger}}{\vec{F}} = \vec{\tau} \leftarrow \text{middle finger}$$

thumb, index finger $\Rightarrow$ middle finger along $\hat{k}$

By physics intuition: $m\vec{g}$ pulls m towards $\vartheta = 0$

$\triangleq$ ccw rotation = math. pos. sense; $\tau_z > 0$

By the rule: φ = angle from $\vec{L}$ towards $m\vec{g}$ $\Rightarrow$ $\sin\varphi = \sin(-\vartheta) = -\sin\vartheta$

$\therefore$ $\underline{M L^2 \vartheta''(t) = - M g L \sin\vartheta(t)}$

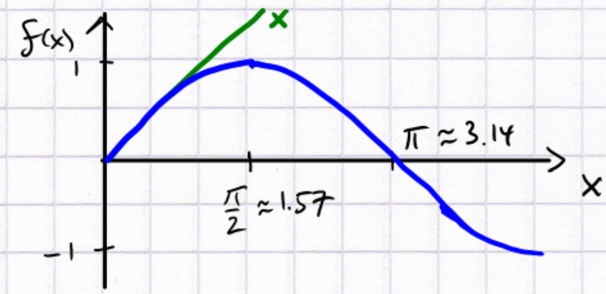
Exercise: draw the mass for $\vartheta > 0$ (on the right) and show: $\tau_z < 0$ in that case!

We have: $\alpha = \frac{d\omega}{dt} = \frac{d^2\vartheta}{dt^2} = -\frac{g}{L}\sin\vartheta(t)$

We cannot find a simple function $\vartheta(t)$ that satisfies Newton's 2^{nd} law $\rightarrow$ choose the small-ϑ approximation.

Idea: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \dots$ Taylor series

Small x: $\sin x \approx x$ is justified



pendulum equation becomes:

$$\vartheta''(t) = -\frac{g}{L}\vartheta(t) \quad \rightarrow \text{ looks like } \quad x''(t) = -\frac{k}{m}x(t)$$

harmonic oscillator

$$\therefore \quad \vartheta(t) = A\cos\left(\sqrt{\tfrac{g}{L}}\,t\right) + B\sin\left(\sqrt{\tfrac{g}{L}}\,t\right) \qquad \omega = \sqrt{\tfrac{k}{m}} \text{ - circ. freq.}$$

↳ clever: "projectile" motion $\{x(t), y(t)\}$ is parametrized by a single variable, polar angle $\vartheta(t)$.

Why does this work? The other freedom = distance from the pivot point O is fixed (pendulum arm L)

Message: for small-ϑ motion (small amplitude) the pendulum has a period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$ which doesn't depend on the amplitude

Application:
- timer for mantle piece clock
- measure g by timing pendulum with known L.

Generalization: "physical" pendulum $\rightarrow$ take inertia of the rod into account.

Mathematical detail from Calculus

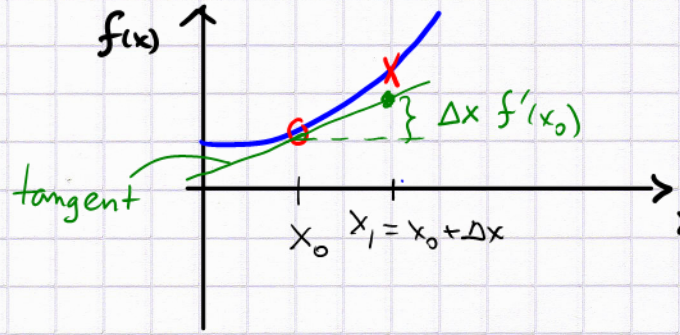
Taylor series. Q: Suppose we know for some $f(x)$

the values $f(x_0)$, $f'(x_0)$, $f''(x_0)$, ...

$\uparrow$ value $\uparrow$ slope $\uparrow$ curvature

can we predict $f(x_1)$ where $x_1 = x_0 + \Delta x$?

$$f(x_1) = f(x_0) + \Delta x \, f'(x_0) + \frac{\Delta x^2}{2!} f''(x_0) + \ldots$$



will this converge for all x ? $\leftarrow$ math question

how well does it work with a few terms ? $\leftarrow$ practical physics question

the meaning ? $\rightarrow$ given $f(x)$, and an expansion point x_0, and some order N, is there a polynomial in $\Delta x \equiv (x - x_0)$ of order N that approximates $f(x)$?

$\rightarrow$ the coefficients of the polynomial are determined by matching : $f(x_0)$, $f'(x_0)$, ... $f^{(N-1)}(x_0)$

Why is this useful ?

1) computer arithmetic can generate $f(x)$ values for complicated functions ; use many x_0 expansion points!

2) complicated functions can be replaced by simple polynomials, but $\Delta x = x - x_0$ has to be small.