Damped oscillations

The harmonic oscillator solution to the second-law statement

$$m\,x''(t) = -k\,x(t)$$

which we also found in the pendulum problem

$$\vartheta'' = \left[-\frac{g}{L}\sin\vartheta \longrightarrow\right] -\frac{g}{L}\,\vartheta(t)$$

small-ϑ approximation

was shown to conserve mechanical energy

$$E = \frac{1}{2}m v^2 + \frac{1}{2}k x^2 = E_0$$

A real oscillator (or pendulum) does not behave this way.

The simplest model is to include linear air drag (which is appropriate for slow motion):

$$m\,x'' = -k\,x(t) \underline{-b\,v(t)}$$

opposes $v(t)$
if $v<0$ it points to right
if $v>0$ it points to left

$$x'' = -\frac{k}{m}x(t) - \frac{b}{m}x'(t)$$

Q: can we guess how the previously found solution

$$x_{HO}(t) = A\cos\omega t + B\sin\omega t \qquad \omega^2 = \frac{k}{m}$$

is modified assuming b is a small constant?

Guess: it should involve multiplying $x_{HO}(t)$ by an exponential function.

Why? $f(t) = e^t$ has the property $f' = f$

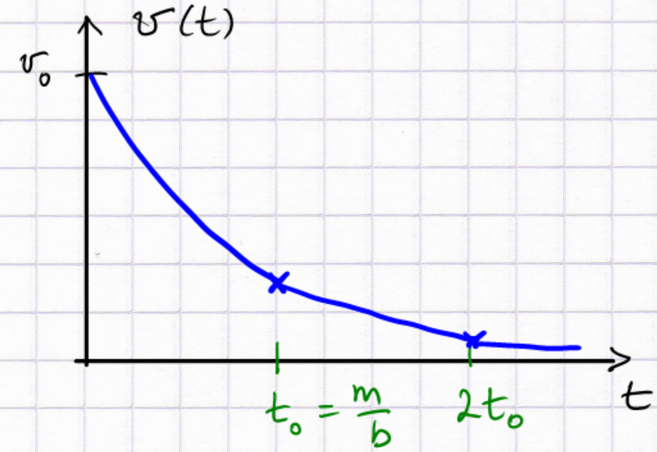
Look at the damping problem by itself (no spring)

$$m x'' = -b x'$$

$$m v' = -b v \qquad v' = -\frac{b}{m} v$$

$$\therefore \quad v(t) = v_0 \, e^{-\frac{b}{m}t} \quad \longrightarrow \quad \frac{dv}{dt} = v_0 \, e^{-\frac{b}{m}t} \underbrace{\left(-\frac{b}{m}\right)}_{\text{chain rule!}}$$

what does this look like?



$t_0 = \frac{m}{b}$ = damping time

in this time v drops by

$$\frac{1}{e} \approx \frac{1}{3} \quad \left(= \frac{1}{2.71\ldots}\right)$$

Euler's nr. defined as a limit of a sequence in Calc.

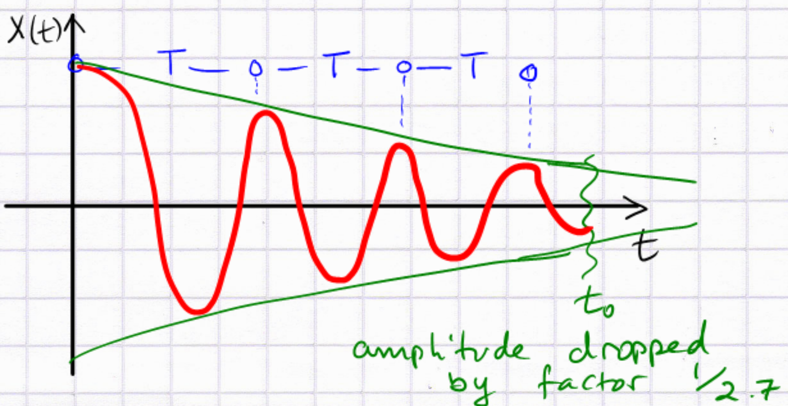
The damped HO eq'n $\quad x'' = -kx - bx'$

is solved by $\quad x(t) = e^{-t/t_0}\left(A \cos \omega_0 t + B \sin \omega_0 t\right)$

where we expect $\quad \omega_0^2 \approx \frac{k}{m} \quad$ and $\quad t_0 = \frac{m}{b}$?

actually $\frac{2m}{b}$

The math is left to PHYS 2010.



oscillation time constant:

$$\omega_0 = 2\pi f = \frac{2\pi}{T} \quad \text{"period"}$$

(not really periodic!)

damping time constant

$$\frac{2m}{b} = t_0$$

So far, we assumed the damping to be weak.

What does this mean?

$$t_0 = \frac{2m}{b} \gg T = \frac{2\pi}{\omega_0} \approx 2\pi\sqrt{\frac{m}{k}}$$

What happens if the damping is increased?

→ non-oscillatory response → exponential behaviour dominates

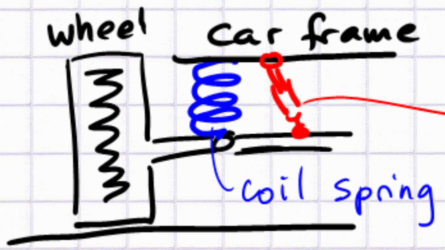
- very large $\frac{m}{b}$ as compared to $\sqrt{\frac{m}{k}}$ →

  mass takes forever to crawl towards equilibrium

- special case: critical damping $t_0 = \frac{T}{2\pi}$

  no oscillation, fastest-possible return to equilibrium.

Application: spring / shock absorber suspension system



shock absorber = cylinder w. piston, but piston has tiny holes; cylinder filled with gas (or oil) which has to pass through → damping

Another application: self-closing door

→ spring mechanism + adjustable friction

friction too low → underdamped → door crashes into lock

" too high → overdamped → door never latches, crawls towards closing