Dynamics

solving Newton's 2nd law, x-motion alone

$a_x = \frac{F_x}{m}$

idea: F_x depends on position, or is constant

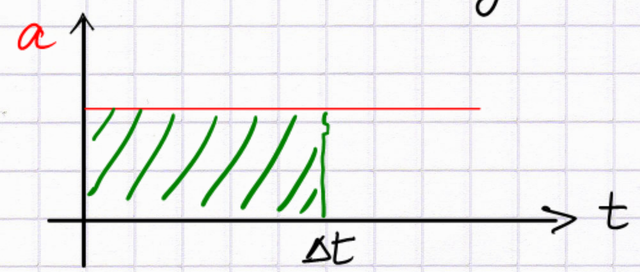
$a_x \to a$, $F_x \to F$

$\longrightarrow$ solve for velocity

- suppose F is constant, then a is constant in time

  constant $\downarrow$ position/time independent, e.g., free fall: $\vec{W} = m\vec{g}$

- Then from kinematics:



$$v_f = v_0 + a\,t_f$$

$$\Delta v = v_f - v_0 = a\,\Delta t$$

why is $a\,\Delta t$ the area?

$\rightarrow$ rectangle: $\boxed{\;}\,a$, Δt

why is it the height (above v_0) in the $v(t)$ graph?

$\rightarrow$ the curve $v(\Delta t) = v_0 + a\,\Delta t$

$$v_f = v_0 + a\,t_f$$

Always use $t_0 = 0$ as a start time for motion,

$\Delta t = t_f - t_0$, or Δt = time of observation

Summary: The definition $a = \frac{dv}{dt}$ leads to the "inverted" statement $\Delta v = a\,\Delta t$, i.e.,

- velocity change is given by the area under $a(t)$

Formal symbol : $\Delta v = v_f - v_0 = \underbrace{\int_0^{t_f} a(t)\, dt}_{\text{holds for general } a(t)!}$ $\quad t_f = \Delta t$, $t_i = 0$

In integral calculus define for any $f(t)$ the anti-derivative $F(t)$.

Example : $f(t) = a \longrightarrow F(t) = at + C$

why can we add a constant ?

$F'(t) = a + 0 = a$

another example : $f(t) = \alpha t \longrightarrow F(t) = \frac{1}{2}\alpha t^2 + C$

why ? $F'(t) = \frac{1}{2}\alpha \cdot 2t + 0 = \alpha t$ ✓

Area under $f(t)$ curve between t_i and t_f :

$$A = \int_{t_i}^{t_f} f(t)\, dt = F(t_f) - F(t_i)$$

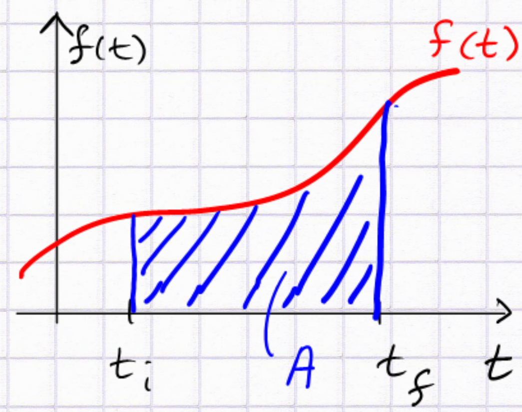


Table of $\{f, F\}$ pairs :

f	F	
0	C	easy!
1	$t + C$	
t	$\frac{1}{2}t^2 + C$	
t^2	$\frac{1}{3}t^3 + C$	
$\cos(t)$	$\sin(t)$	← hard to show
$\exp(t)$	$\exp(t)$	

math application :

prove that

$A_{circle} = \pi R_c^2$

$V_{sphere} = \frac{4}{3}\pi R_s^3$

$A_{sphere} = 4\pi R_s^2$

etc.

Back to physics:

given $a(t)$, math rules allow to obtain $\Delta v(t)$ as an area under the curve.

How do we get the displacement $\Delta x(t) = x_f(t) - x_0$?

Area under the $v(t)$ curve!

$$v(t) \equiv \frac{dx}{dt} \Rightarrow \Delta x = x_f - x_0 = \int_0^{t_f} v(t)\,dt \qquad (\Delta t = t - 0)$$

Apply this to constant-force case:

$$a = \frac{F}{m} \quad \therefore \quad \frac{dv}{dt} = \frac{F}{m} = a \quad (= \text{const})$$

$$\Delta v = v_f - v_0 = \int_0^{t_f} a\,dt = a t_f$$

$$v_f = v_0 + a t_f \quad \therefore \ \text{generalize to any } t_f \quad v(t) = v_0 + at$$

$$\Delta x = x_f - x_0 = \int_0^{t_f} v(t)\,dt = \int_0^{t_f} \underbrace{(v_0 + at)}_{f(t)}\,dt = \underbrace{\left[v_0 t + \tfrac{1}{2} a t^2\right]}_{F(t)}{}_0^{t_f}$$

$$= v_0 t_f - 0 + \frac{1}{2} a t_f^2 - 0$$

$$\therefore \quad x_f = x_0 + v_0 t_f + \frac{1}{2} a t_f^2 \qquad \text{now choose arbitrary } t_f \to t$$

$$\boxed{x(t) = x_0 + v_0 t + \frac{1}{2} a t^2}$$

(A) We had two integration steps ($a \to \Delta v$ and $v \to \Delta x$)

Each introduces an undetermined constant, namely v_0 and x_0.

(MATH statement)

(B) when setting up the motion of an object there are two freedoms:

1) initial position x_0; 2) initial velocity v_0

(PHYS statement)