# DISCUSSION OF "ESTIMATION OF MULTI-DIMENSIONAL LOG-CONCAVE DENSITY" BY CULE, SAMWORTH, AND STEWART

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I would like to congratulate the authors on an important and thought-provoking paper. This work will certainly be a catalyst for further research in the area of shape-constrained estimation, and the authors themselves suggest several open problems towards the end of the paper. I will restrict my discussion to adding another question to this list.

One of the identifying features of nonparametric shape-constrained estimators are their rates of convergence, which are slower than the typical $n^{1/2}$ rate achieved by parametric estimators. In one dimension, the Grenander estimator of the decreasing density converges at a local rate of $n^{1/3}$ while the estimator of a convex decreasing density converges locally at rate $n^{2/5}$ [PR, Gro2, GJW]. A similar rate is seen for the one dimensional nonparametric maximum likelihood estimator (NPMLE) of a log-concave density, which was recently proved to be $n^{2/5}$, as long as the density is strictly log-concave [BRW]. A heuristic justification of how different local rates arise is given in [KP]. The global convergence rates, on the other hand, can be quite different. For the Grenander estimator, the convergence rate for functionals $\varphi(\widehat{f}_n - f_0)$ is known to be

$$n^{1/6} \left\{ n^{1/3} \varphi(\widehat{f}_n - f_0) - \mu_\varphi(f_0) \right\} \Rightarrow \sigma_\varphi(f_0) Z,$$

where $Z$ is a standard normal random variable [Gro1, GHL, KL]. Here, $f_0$ denotes the true underlying monotone density. Thus, smooth functionals with $\mu_\varphi(f_0) = 0$ (such as plug-in estimators of the moments) converge at rate $n^{1/2}$ and recover the faster rate characteristic of parametric estimators.

Global and local convergence rates for the log-concave NPMLE are sure to be of much interest in the near future. Indeed, it is already conjectured in [SW] that the local convergence rate for the estimator $\widehat{f}_n$ introduced here is $n^{2/(4+d)}$ when $d = 2, 3$. In Section 7, the authors consider plug-in estimators of the moments or the differential entropy for $\widehat{f}_n$. What would the convergence rate be for these functionals? Preliminary simulations for $d = 1$ indicate that the $n^{1/2}$ rate may continue to hold for the log-concave maximum likelihood estimators (see Figure 1). Further investigation is needed in higher dimensions. A rate of $n^{1/2}$ would, naturally, be very attractive in the application of these methods.

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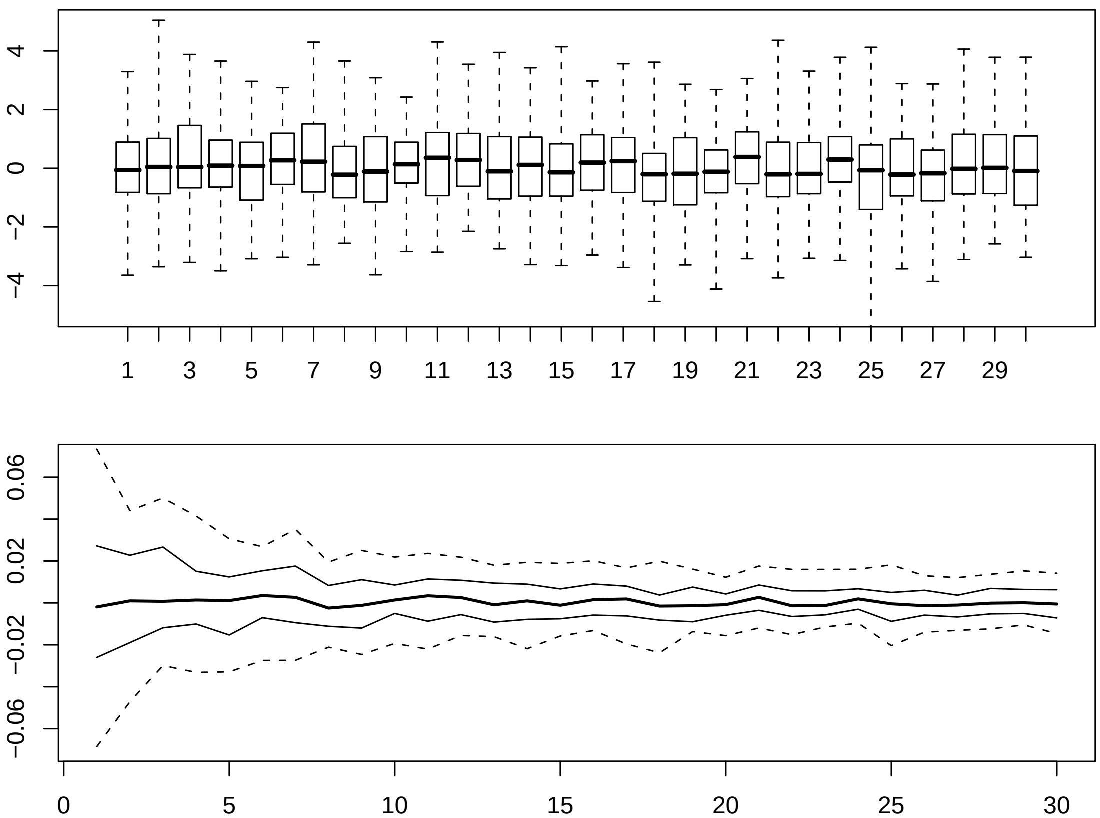


FIGURE 1. (Top) $n^{1/2}$ re-scaled functional vs. sample size (in 1000s): the NPMLE of a Gamma(2,1) random variable was computed (using [RD]) and the centered mean functional was calculated based on the estimated density. Each boxplot consists of $B = 100$ simulations. (Bottom) Quantiles vs. sample size (in 1000s): quantiles of the un-scaled and centered functionals (0.05 and 0.95 dashed, 0.25 and 0.75 solid, median in bold). A regression of the logarithm on the 0.05 and 0.95 quantiles on the logarithm of sample size yields a highly significant slope estimate of $-0.48968$.

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